1.12 Applications of Rational Equations

In this lesson, we will explore some of the applications that involve rational equations, specifically motion related problems and work related problems. The mathematical models for these types of problems are similar, but not identical, which may help students to remember the models better. We will begin by introducing the formulas for both models, and then we will consider the models separately, beginning with the work related model.

Formulas:

Motion	$d = rt$	distance = rate x time
Work	$w = rt$	work = rate x time

These types of word problems can be organized in a table which makes it easier to solve.

<u>Work Examples</u>

1. A homeowner estimates that it will take him 8 days to renovate his back yard. A professional landscaper estimates that he could finish the job in 5 days. How long will it take if the homeowner helps the landscaper?

	rate (per day)	**(together) time**	**work done = rate x time (per day)**
homeowner	$\frac{1}{8}$	t	$\frac{1}{8}t = \frac{t}{8}$
roofer	$\frac{1}{5}$	t	$\frac{1}{5}t = \frac{t}{5}$

If it takes the homeowner 8 days to complete 1 job, then the homeowner completes $\frac{1}{8}$ of the job in one day. Likewise, the landscaper completes $\frac{1}{5}$ of the job in one day.

If we combine (add) the work done by the homeowner and the landscaper, it equals one job completed.

$\frac{t}{8} + \frac{t}{5} = 1$ Solve the rational equation for t. LCM$= 40$.

$\frac{5}{5} \cdot \frac{t}{8} + \frac{t}{5} \cdot \frac{8}{8} = \frac{1}{1} \cdot \frac{40}{40}$ Simplify.

$\frac{5t}{40} + \frac{8t}{40} = \frac{40}{40}$ Clear fractions.

$5t + 8t = 40$ Solve for t.

$$13t = 40$$

$$\frac{\cancel{13}t}{\cancel{13}} = \frac{40}{13}$$

$t = \frac{40}{13} = 3\frac{1}{13}$ days

2. In 10 minutes, a machine can sort a large barrel of recyclables. A smaller machine can sort the same amount of recyclables in 16 minutes. If both machines are used, how long will it take to sort a large barrel of recyclables?

	rate (per minute)	**(together) time**	**work done = rate x time (per minute)**
larger machine	$\frac{1}{10}$	t	$\frac{1}{10}t = \frac{t}{10}$
smaller machine	$\frac{1}{16}$	t	$\frac{1}{16}t = \frac{t}{16}$

If it takes the larger machine 10 minutes to complete 1 job, then the larger machine completes $\frac{1}{10}$ of the job in one minute. Likewise, the smaller machine completes $\frac{1}{16}$ of the job in one minute.

If we combine (add) the work done by the two machines, it equals one job completed.

$$\frac{t}{10}+\frac{t}{16}=1$$ Solve the rational equation for t. LCM$=80$.

$$\frac{8}{8}\cdot\frac{t}{10}+\frac{t}{16}\cdot\frac{5}{5}=\frac{1}{1}\cdot\frac{80}{80}$$ Simplify.

$$\frac{8t}{80}+\frac{5t}{80}=\frac{80}{80}$$ Clear fractions.

$$8t+5t=80$$ Solve for t.

$$13t=80$$

$$\frac{13t}{13}=\frac{80}{13}$$

$$t=\frac{80}{13}=6\frac{2}{13}\text{ minutes}$$

3. One hose can fill a small pool in 3 days and a second smaller hose can fill it in 5 days. However, evaporation can empty the pool in 120 days. If both hoses are used, how long will it take to fill the pool?

	rate (per week)	**(together) time**	**work done = rate x time (per week)**
first hose	$\frac{1}{3}$	t	$\frac{1}{3}t=\frac{t}{3}$
second hose	$\frac{1}{5}$	t	$\frac{1}{5}t=\frac{t}{5}$
evaporation	$\frac{-1}{120}$	t	$\frac{-1}{120}t=\frac{-t}{120}$

If it takes the first hose 3 days to complete 1 job (filling pond), then the first hose completes $\frac{1}{3}$ of the job in one day. Likewise, the second hose completes $\frac{1}{5}$ of the job (filling pool) in one day and the evaporation completes $\frac{-1}{120}$ of the job (emptying pool) in one day.

$$\frac{t}{3}+\frac{t}{5}-\frac{t}{120}=1$$ Solve for *t*. LCM$=120$.

$$\frac{40}{40}\cdot\frac{t}{3}+\frac{t}{5}\cdot\frac{24}{24}-\frac{t}{120}=\frac{1}{1}\cdot\frac{120}{120}$$ Simplify.

$$\frac{40t}{120}+\frac{24t}{120}-\frac{t}{120}=\frac{120}{120}$$ Clear fractions.

$$40t+24t-t=120$$ Solve for *t*.

$$63t=120$$

$$\frac{63t}{63}=\frac{120}{63}$$

$$t=\frac{120}{63}=1\frac{57}{63}\text{ days.}$$

4. It takes one team 4 hours to set up for an event. If a second team helps, it only takes 3 hours. How long would it take the second team, working alone, to set up for the event?

We know it takes the first team 4 hours to set up working alone. Since we do not know how long it will take the second team alone, we can let *r* = time for the second team to set up working alone. Now, we can complete the table knowing that it takes the two teams 3 hours to set up if they are working together.

	rate (per hour)	(together) time	work done = rate x time (per hour)
first crew	$\frac{1}{4}$	3	$\frac{1}{4} \cdot 3 = \frac{3}{4}$
second crew	$\frac{1}{r}$	3	$\frac{1}{r} \cdot 3 = \frac{3}{r}$

If we combine (add) the work done by both teams, it equals one job completed.

$\frac{3}{4} + \frac{3}{r} = 1$ Solve for r. LCM$= 4r$.

$\frac{r}{r} \cdot \frac{3}{4} + \frac{3}{r} \cdot \frac{4}{4} = 1 \cdot \frac{4r}{4r}$ Simplify.

$\frac{3r}{4r} + \frac{12}{4r} = \frac{4r}{4r}$ Clear fractions.

$$\begin{array}{rcl} 3r + 12 &=& 4r \\ -3r \qquad && - 3r \\ \hline 12 &=& r \end{array}$$

Solve for r.

$r = 12$ hours

Distance-Rate-Time Examples

1. A plane can fly 600 miles in the same amount of time as it takes a car to go 250 miles. If the plane travels 98 mph faster than the car, find the speed of the car.

For our purposes, speed = rate.
Recall: $d = rt$ (distance = rate x time)

	rate (miles per hour)	**time (hours)**	**distance (miles)**	**distance = rate x time** $\boldsymbol{d = rt}$
plane	$r + 98$	t	600	$(r + 98)t = 600$
car	r	t	250	$rt = 250$

Since we know that the plane and the car both traveled the same amount of time, we can solve each equation from the table for *t* and by setting the equations equal to each other, we can solve.

$$rt = 250$$
$$\frac{rt}{r} = \frac{250}{r}$$
$$t = \frac{250}{r}$$

$$(r + 98)t = 600$$
$$\frac{(r+98)t}{(r+98)} = \frac{600}{r+98}$$
$$t = \frac{600}{r+98}$$

Since both equations are now solved for *t*, we can set the other side of the equations equal to each other and solve for *r* to find the speed of the plane.

$$\frac{250}{r} = \frac{600}{r+98}$$ LCM= $r(r + 98)$

$$\frac{r+98}{r+98} \cdot \frac{250}{r} = \frac{600}{r+98} \cdot \frac{r}{r}$$ Simplify.

$$\frac{250(r+98)}{r(r+98)} = \frac{600r}{r(r+98)}$$ Clear fractions.

$250(r+98) = 600r$ Solve for *r*.

$$\begin{aligned} 250r + 24500 &= 600r \\ -250r \quad\quad &\;\; -250r \\ \hline 24500 &= 350r \end{aligned}$$

$$\frac{24500}{350} = \frac{350r}{350}$$

$r = 70$ The speed of the car is 70 mph.

2. Two cars, a Honda and a Toyota, each travelled 600 miles on a road trip. The Honda travelled 15 mph faster than the Toyota and it arrived 2 hours earlier. Find the speed of each car.

	rate (miles per hour)	**time (hours)**	**distance (miles)**	**distance = rate x time** $\boldsymbol{d = rt}$
Honda (faster)	$r+15$	$t-2$	600	$600 = (r+15)(t-2)$
Toyota	r	t	600	$600 = rt$

Since we want to know the speed of each car, we can solve each equation in the table for *t*.

$$600 = (r+15)(t-2)$$

$$\frac{600}{r+15} = \frac{\cancel{(r+15)}(t-2)}{\cancel{(r+15)}}$$

$$\frac{600}{r+15} = t - 2$$
$$+2 \quad +2$$
$$\frac{600}{r+15} + 2 = t$$

$$600 = rt$$

$$\frac{600}{r} = \frac{rt}{r}$$

$$\frac{600}{r} = t$$

$$t = \frac{600}{r}$$

$t = \frac{600}{r+15} + 2$

Set the equations equal to each other and solve for *r*.

$$\frac{600}{r+15} + 2 = \frac{600}{r} \qquad \text{LCM} = r(r+15)$$

$$\frac{r}{r} \cdot \frac{600}{r+15} + \frac{2}{1} \cdot \frac{r(r+15)}{r(r+15)} = \frac{600}{r} \cdot \frac{r+15}{r+15} \qquad \text{Simplify.}$$

$$\frac{600r}{r(r+15)} + \frac{2r(r+15)}{r(r+15)} = \frac{600(r+15)}{r(r+15)} \qquad \text{Clear fractions.}$$

$$600r + 2r(r+15) = 600(r+15) \qquad \text{Solve for } r.$$

$$600r + 2r^2 + 30r = 600r + 9000$$

$$\begin{array}{rcl} 2r^2 + 630r &=& 600r + 9000 \\ -600r && -600r \\ \hline 2r^2 + 30r &=& 9000 \\ -9000 && -9000 \\ \hline 2r^2 + 30r - 9000 &=& 0 \end{array}$$

$$2(r^2 + 15r - 4500) = 0$$

Even though 4500 is a large number, it can be broken down easily using a factor tree:

$4500 = 45 \cdot 100 = 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 2 \cdot 2$

Then factors can be combined in different ways to try to get the middle term. The products need to be 15 apart, so

$5 \cdot 2 \cdot 2 \cdot 3$ and $5 \cdot 5 \cdot 3$ will work.

$$2(r-60)(r+75) = 0$$

$$\frac{\cancel{2}(r-60)(r+75)}{\cancel{2}} = \frac{0}{2}$$

$$(r-60)(r+75) = 0$$

$$\begin{array}{rcl} r - 60 &=& 0 \\ +60 && +60 \\ \hline r &=& 60 \end{array} \qquad \begin{array}{rcl} r - 75 &=& 0 \\ -75 && -75 \\ \hline r &=& -75 \end{array}$$

This gives two rates, 60 mph and -75 mph. Since the rate/speed must be positive, we can "drop" the rate -75 mph (this is extraneous). From the table above, 60 mph represents the rate/speed of the Toyota. To find the rate of the Honda, we substitute 60 into $r + 15$ to get 75 mph.

3. The speed of a river current is 7 mph. If a boat travels 20 miles downstream in the same time that it takes to travel 10 miles upstream, find the speed of the boat in still water.

The boat travels upstream and downstream the same amount of time, but travels further downstream than it does upstream. This means the rate of the boat travelling upstream is slower than the rate traveling downstream. So, we adjust the rate, r, by adding/subtracting the rate of the current which is 7 mph.

	rate (miles per hour)	**time (hours)**	**distance (miles)**	**distance = rate x time** $\boldsymbol{d = rt}$
upstream	$r - 7$	t	10	$10 = (r-7)t$
downstream	$r + 7$	t	20	$20 = (r+7)t$

Solve the two equations in the table for *t*.

$$10 = (r-7)t$$
$$\frac{10}{r-7} = \frac{\cancel{(r-7)}t}{\cancel{r-7}}$$
$$t = \frac{10}{r-7}$$

$$20 = (r+7)t$$
$$\frac{20}{r+7} = \frac{\cancel{(r+7)}t}{\cancel{r+7}}$$
$$t = \frac{20}{r+7}$$

Since both equations are in terms of *t*, we can set the equations equal to each other and solve for *r*.

$$\frac{10}{r-7} = \frac{20}{r+7}$$

LCM= $(r-7)(r+7)$

$$\frac{r+7}{r+7}\cdot\frac{10}{r-7} = \frac{20}{r+7}\cdot\frac{r-7}{r-7}$$

Simplify and clear fractions.

$$\begin{aligned} 10r + 70 &= 20r - 140 \\ -10r \quad &\quad - 10r \\ \hline 70 &= 10r - 140 \\ +140 \quad &\quad + 140 \\ \hline 210 &= 10r \end{aligned}$$

Solve for r.

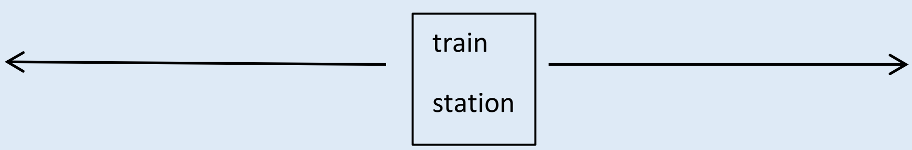
$$\frac{210}{10} = \frac{10r}{10}$$

$$21 = r$$

The rate of the boat in still water is 21 mph.

4. Two trains going in opposite directions leave at the same time. Train 2 travels 10 mph faster than Train 1. In 6 hours, the trains are 670 miles apart. Find the speed of each train.

Sometimes it is helpful to draw a picture.



$$\underbrace{\hspace{6cm}}_{}$$

$$distance\ after\ 6\ hours = 670\ miles$$

	rate (miles per hour)	time (hours)	distance (miles)	distance = rate x time $\boldsymbol{d = rt}$
train 1	r	6	$6r$	$d_1 = 6r$
train 2	$r + 10$	6	$6(r+10)$	$d_2 = 6(r+10)$

We can see from the diagram above, that the combined distance of the trains is 300 miles. So,

$$d_1 + d_2 \rightarrow 6r + 6(r + 10) = 670$$

Solve for r.

$$6r + 6r + 10 \;\; = 670$$

$$12r + 10 = 670$$

$$\underline{\quad - 10 \quad - 10}$$

$$12r \;\; = 660$$

$$\frac{12r}{12} = \frac{660}{12}$$

$$r = 55$$

Train 1 $= 55$ mph and Train 2$= 55 + 10 = 65$ mph.