1.11 Solving Rational Equations

Now we are back to solving equations, which was one of the major reasons we learned to work with rational expressions. Rational equations occur in distance-rate-time problems and in other types of work related problems as you will see in the next section. In order to solve a rational equation, you will need the techniques used to solve polynomial equations because you can reduce each rational equation to a polynomial equation by eliminating the denominator. There are two ways to think about this. You can either write each side with the common denominator and then use the logic that says if the denominators are equal, then the numerators must also be equal, or you can multiply both sides of the equation by the common denominator, in which case you will arrive at the same polynomial equation obtained by thinking of it the first way. Either way, you arrive at a polynomial equation and once you solve it, you must make sure that your answers are valid for the original rational equation. If any of your answers make any denominator equal 0, then you must throw that answer away. (It may solve the derived polynomial equation, but it does not solve the original rational equation.) When this happens, we call the solution '*extraneous*'. Any solution to the polynomial equation that does not make any denominator equal 0 is also a solution of the rational equation.

Examples

1. $$\frac{3}{4}+\frac{5}{8}=\frac{x}{12}$$ Get a common denominator.

LCD$= 24$.

$$\frac{6}{6}\cdot\frac{3}{4}+\frac{5}{8}\cdot\frac{3}{3}=\frac{x}{12}\cdot\frac{2}{2}$$ Simplify.

$$\frac{18}{24}+\frac{15}{24}=\frac{2x}{24}$$ To get rid of the denominator, multiply both sides by LCD.

$$\frac{\cancel{24}}{1}\left(\frac{18+15}{\cancel{24}}\right) = \frac{2x}{\cancel{24}} \cdot \frac{\cancel{24}}{1}$$ Simplify.

$$33 = 2x$$ Solve for *x*.

$$\frac{33}{2} = \frac{\cancel{2}x}{\cancel{2}}$$

$$x = \frac{33}{2} \text{ or } x = 16\tfrac{1}{2} \text{ or } x = 16.5$$

2. $$\frac{-5}{x} + \frac{x}{2} = \frac{3}{x}$$ Get a common denominator.

LCD = $2x$.

$$\frac{2}{2} \cdot \frac{-5}{x} + \frac{x}{2} \cdot \frac{x}{x} = \frac{3}{x} \cdot \frac{2}{2}$$ Simplify.

$$\frac{-10}{2x} + \frac{x^2}{2x} = \frac{6}{2x}$$ To get rid of the denominator, multiply both sides by LCD.

$$\frac{\cancel{2x}}{1} \cdot \left(\frac{-10+x^2}{\cancel{2x}}\right) = \frac{6}{\cancel{2x}} \cdot \frac{\cancel{2x}}{1}$$ Simplify.

$$-10 + x^2 = 6$$ Solve for *x*.

There are 2 possible ways to solve for x:

<u>Method 1 (Factor):</u>

$$-10 + x^2 = 6$$
$$\underline{\;-6 \qquad\quad -6\;}$$
$$-16 + x^2 = 0$$

$$x^2 - 16 = 0$$

<u>Method 2: (Isolate the square)</u>

$$-10 + x^2 = 6$$
$$\underline{+10 \qquad\quad +10}$$
$$x^2 = 16$$

$$\sqrt{x^2} = \pm\sqrt{16}$$

$$(x+4)(x-4)=0 \qquad x = \pm 4$$

$$\begin{array}{rcl} x+4=0 & & x-4=0 \\ \underline{\;-4\;\;-4\;} & & \underline{\;+4\;\;+4\;} \\ x=-4 & & x=4 \end{array}$$

As you can see, the results are the same regardless of which method you use.

You must check to make sure that the potential solutions do not make the denominator 0. Since ± 4 does not make any denominator in the original problem 0, then both values are solutions of the rational equation.

3. $x + \frac{1}{x} = 2$ Get common denominator. LCD$= x$.

$\frac{x}{x} \cdot \frac{x}{1} + \frac{1}{x} = \frac{2}{1} \cdot \frac{x}{x}$ Simplify.

$\frac{x^2+1}{x} = \frac{2x}{x}$ Multiply by *x* to clear fractions.

$\frac{\cancel{x}}{1} \cdot \frac{x^2+1}{\cancel{x}} = \frac{2x}{\cancel{x}} \cdot \frac{\cancel{x}}{1}$ Simplify.

$$\begin{array}{r} x^2+1=2x \\ \underline{\;-2x\quad -2x\;} \\ x^2-2x+1=0 \end{array}$$

Get 0 on one side.

Factor.

$(x-1)(x-1)=0$ Set the factor equal to 0, then solve for *x*.

$$\begin{array}{r} x-1=0 \\ \underline{\;+1\;\;+1\;} \\ x=1 \end{array}$$

Since this potential answer does not make the denominator equal to 0, it is a valid solution of the rational equation.

4. $\frac{4}{x-2} + \frac{1}{x+2} = \frac{26}{x^2-4}$ Factor each denominator, if possible.

$\frac{4}{x-2} + \frac{1}{x+2} = \frac{26}{(x+2)(x-2)}$ LCD = $(x+2)(x-2)$

$\frac{(x+2)}{(x+2)} \cdot \frac{4}{x-2} + \frac{1}{x+2} \cdot \frac{(x-2)}{x-2} = \frac{26}{(x+2)(x-2)}$ Simplify.

$\frac{4(x+2)}{(x+2)(x-2)} + \frac{1(x-2)}{(x+2)(x-2)} = \frac{26}{(x+2)(x-2)}$ Clear fractions.

$4x + 8 + x - 2 = 26$ Simplify.

$$\begin{array}{rl} 5x + 6 &= 26 \\ -6 & -6 \\ \hline 5x &= 20 \end{array}$$

Solve for x.

$$\frac{5x}{5} = \frac{20}{5}$$

$x = 4$ Check potential solution.
This is a valid solution.

5. $\frac{5}{y-3} - \frac{30}{y^2-9} = 1$ Factor denominators.

$\frac{5}{y-3} - \frac{30}{(y+3)(y-3)} = 1$ LCD = $(y+3)(y-3)$.

$\frac{y+3}{y+3} \cdot \frac{5}{y-3} - \frac{30}{(y+3)(y-3)} = 1 \cdot \frac{(y+3)(y-3)}{(y+3)(y-3)}$ Simplify.

$$\frac{5(y+3)}{(y+3)(y-3)} - \frac{30}{(y+3)(y-3)} = \frac{(y+3)(y-3)}{(y+3)(y-3)}$$ Clear fractions.

$5y + 15 - 30 = y^2 - 9$ Simplify.

$$\begin{array}{rl} 5y - 15 & = y^2 - 9 \\ -5y + 15 & \quad - 5y + 15 \\ \hline 0 & = y^2 - 5y + 6 \\ 0 & = (y-2)(y-3) \end{array}$$ Solve for y.

$$\begin{array}{rl} y - 2 & = 0 \\ +2 & \;\; +2 \\ \hline y & = 2 \end{array} \qquad \begin{array}{rl} y - 3 & = 0 \\ +3 & \;\; +3 \\ \hline \cancel{y} & \cancel{= 3} \end{array}$$

Since the denominator is 0 when you check 3, this potential solution is extraneous. So, the only valid solution is 2.

6. $$\frac{x}{x+4} - \frac{4}{x-4} = \frac{x^2+16}{x^2-16}$$ Factor denominators.

$$\frac{x}{x+4} - \frac{4}{x-4} = \frac{x^2+16}{(x+4)(x-4)}$$ LCD= $(x+4)(x-4)$.

$$\frac{x-4}{x-4} \cdot \frac{x}{x+4} - \frac{4}{x-4} \cdot \frac{x+4}{x+5} = \frac{x^2+16}{(x+4)(x-4)}$$ Simplify and clear fractions.

$x(x-4) - 4(x+4) = x^2 + 16$ Distribute.

$x^2 - 4x - 4x - 16 = x^2 + 16$ Simplify.

$$
\begin{array}{l}
x^2 - 8x - 16 = x^2 + 16 \\
\underline{-x^2 \qquad\qquad\quad -x^2} \\
-8x - 16 = 16 \\
\underline{\quad +16 \quad +16} \\
-8x = 32
\end{array}
$$

Solve for x.

$$\frac{-8x}{-8} = \frac{32}{-8}$$

$$x = -4$$

This is the proposed solution that we must check to make sure it is valid. Since this makes a denominator 0, it is not a valid solution. So, there is no solution to this rational equation.

7. $$\frac{x+13}{2x} + 2 = \frac{3(x+1)}{x^2}$$

LCD= $2x^2$

$$\frac{x}{x} \cdot \frac{x+13}{2x} + \frac{2}{1} \cdot \frac{2x^2}{2x^2} = \frac{3(x+1)}{x^2} \cdot \frac{2}{2}$$

Simplify and clear fractions.

$$x^2 + 13x + 4x^2 = 6x + 6$$

$$
\begin{array}{l}
5x^2 + 13x \qquad = 6x + 6 \\
\underline{\qquad -6x - 6 \quad -6x - 6} \\
5x^2 + 7x - 6 = 0
\end{array}
$$

Solve polynomial equation.

$$(5x - 3)(x + 2) = 0$$

$$5x - 3 = 0 \qquad x + 2 = 0$$
$$\underline{\quad +3 \quad +3} \qquad \underline{-2 \quad -2}$$
$$5x = 3 \qquad x = -2$$
$$\frac{5x}{5} = \frac{3}{5}$$
$$x = \frac{3}{5}$$

After checking, these are both valid solutions.

8. $$\frac{x-4}{x-3} - \frac{x-2}{3-x} = \frac{x-3}{1}$$

If you look at the denominators, you can see that if -1 is factored from $3 - x$ you will get $-(x-3)$, so the LCD= $x - 3$. Note: The negative sign is moved in front of the fraction (the double negative becomes a +).

$$\frac{x-4}{x-3} - \frac{x-2}{-(x-3)} = \frac{x-3}{1}$$

$$\frac{x-4}{x-3} - -\frac{x-2}{x-3} = \frac{x-3}{1}$$
LCD $= x - 3$.

$$\frac{x-4}{x-3} + \frac{x-2}{x-3} = \frac{x-3}{1} \cdot \frac{x-3}{x-3}$$

Simplify and clear fractions.

$$x - 4 + x - 2 = (x-3)(x-3)$$

Simplify and solve for x.

$$2x - 6 = x^2 - 6x + 9$$
$$\underline{-2x + 6 \qquad -2x + 6}$$
$$0 = x^2 - 8x + 15$$
$$0 = (x-3)(x-5)$$

Solve polynomial equation.

$$x - 3 = 0 \qquad x - 5 = 0$$
$$\underline{+3 \quad +3} \qquad \underline{+5 \quad +5}$$
$$x = 3 \qquad x = 5$$

After checking solutions, $x = 3$ is determined to be extraneous. So, the only valid solution is $x = 5$.

9. $y^{-2} + y^{-1} = 2$ — Rewrite with positive exponents.

$$\frac{1}{y^2} + \frac{1}{y} = 2$$ LCD = y^2

$$\frac{1}{y^2} + \frac{1}{y} \cdot \frac{y}{y} = \frac{2}{1} \cdot \frac{y^2}{y^2}$$ Simplify and clear fraction.

$$\begin{array}{rl} 1 + y &= 2y^2 \\ -1 - y & \quad - y - 1 \\ \hline 0 &= 2y^2 - y - 1 \\ 0 &= (2y+1)(y-1) \end{array}$$ Solve polynomial equation.

$$\begin{array}{rl} 2y + 1 &= 0 \\ -1 & \; -1 \\ \hline 2y &= -1 \\ \frac{2y}{2} &= \frac{-1}{2} \\ y &= \frac{-1}{2} \end{array} \qquad \begin{array}{rl} y - 1 &= 0 \\ +1 & \; +1 \\ \hline y &= 1 \end{array}$$

Both solutions are valid.

The remaining examples are literal equations. When solving literal equations, isolate the variable you are solving for and just treat the other variables like numbers.

Note: Lower case and capital letters are considered different variables.

10. $\frac{E}{e} = \frac{R+r}{r}$; solve for r. — We want to isolate r. So, get a common denominator. LCD= er.

$$\frac{r}{r} \cdot \frac{E}{e} = \frac{R+r}{r} \cdot \frac{e}{e}$$ Simplify.

$$\frac{rE}{re} = \frac{e(R+r)}{re}$$ Clear fractions.

$\frac{\cancel{re}}{1} \cdot \frac{rE}{\cancel{re}} = \frac{e(R+r)}{\cancel{re}} \cdot \frac{\cancel{re}}{1}$ Simplify.

$rE = eR + er$ To isolate r, get all terms with r on one side.

$\underline{\quad -er \qquad -er}$

$rE - er = eR$ Factor out r.

$r(E - e) = eR$ Divide by $E - e$ on both sides of equation.

$\frac{r(\cancel{E-e})}{\cancel{E-e}} = \frac{eR}{E-e}$

$r = \frac{eR}{E-e}$

11. $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$; solve for x. Get common denominator.

LCD = xyz.

$\frac{yz}{yz} \cdot \frac{1}{x} = \frac{1}{y} \cdot \frac{xz}{xz} + \frac{1}{z} \cdot \frac{xy}{xy}$ Simplify.

$\frac{yz}{xyz} = \frac{xz+xy}{xyz}$ Clear fraction.

$yz = xz + xy$ We want to isolate x. So, factor.

$yz = x(z + y)$ Divide by $(x + y)$ on both sides.

$\frac{yz}{z+y} = \frac{x(\cancel{z+y})}{\cancel{z+y}}$

$x = \frac{yz}{z+y}$