1.9 Rational Expressions: Adding and Subtracting

When we add or subtract fractions, what we are really doing is counting pieces of a whole which is divided up in a certain way. If we are to count items, then they must be the same kind of items for our counting to be meaningful. So if we are counting pieces, then they have to be the same size. For example, imagine a pizza that is cut into 8 pieces. If you eat 1 slice, then you have eaten $\frac{1}{8}$ of the whole pizza. Of course, each of these 8 pieces must be the same size in order for us to say that you ate $\frac{1}{8}$ of the pizza! If not, you may have eaten half for all we know! The denominator tells you how many slices there are (or what kind of piece you are dealing with – here we are dealing with eighths) and the numerator tells you "how many" out of those are being included. So if you ate $\frac{1}{8}$ of the pizza, then how much pizza is left? You should arrive at $\frac{7}{8}$ if you think about it, because there are 7 pieces left out of the 8 that we started with. Note that we could think of this as a subtraction problem since 1 whole pizza equals 8 pieces of pizza, or $\frac{8}{8}$, we see that $1-\frac{1}{8}=\frac{8}{8}-\frac{1}{8}=\frac{7}{8}$. This analogy could help you recall the processes in arithmetic for adding and subtracting fractions. We review this process below and then we apply the same process with variables. It is as simple as that!

<u>Arithmetic: Adding/Subtracting Fractions</u>

If both fractions have the same denominator, we can just add (or subtract) the numerators and keep the SAME denominator. This is because we have the same size pieces and we can just go ahead and count them by adding the numerators.

how many?

$$\frac{3}{7}+\frac{2}{7}=\frac{5}{7}$$

size
(kind of piece)

If both fractions **<u>do not</u>** have the same denominator, we must get a common denominator and then add the numerators (keep the SAME denominator). This process is analogous to cutting the pizza into smaller pieces so that we have more pieces in order to make them the same size (the common denominator is the number of pieces we would have). If the pieces are not the same size, it makes no sense to count them!

$$\frac{3}{8}+\frac{7}{12}=\frac{3}{3}\cdot\frac{3}{8}+\frac{7}{12}\cdot\frac{2}{2}=\frac{9}{24}+\frac{14}{24}=\frac{23}{24}$$

To get a common denominator, you must find the least common multiple (LCM) of the denominators. This will be the smallest number that all of the denominators will divide evenly (i.e. with no remainder) For the denominators above, 8 and 12, identify the prime factors.

$8 = 2 \cdot 2 \cdot 2$ $\qquad\qquad$ $12 = 2 \cdot 2 \cdot 3$

Now, count the number of each factor from each denominator choose the most factors from one of the denominators. Repeat for each factor.

There are 3 factors of 2 from 8 and 2 factors of 2 from 12. So, we will choose and use 3 factors of 2.

There are no factors of 3 from 8 and 1 factor of 3 from 12. So, we will choose and use 1 factor of 3.

The least common denominator (LCD) $= 2 \cdot 2 \cdot 2 \cdot 3 = 24$.

Now, multiply each fraction by a form of 1 such as 3/3 or 2/2 to get a common denominator.

We can use the same process with rational expressions that contain variables. Factor the denominators to find the LCD. Then multiply each fraction by what is missing in both the denominator and the numerator. Then add/subtract and simplify (or reduce) your answer if possible.

Examples

1. $\frac{3x}{x^2-4} - \frac{6}{x^2-4}$ Since there is a common denominator, subtract the numerators.

$= \frac{3x-6}{x^2-4}$ Simplify (factor and cancel, if possible).

$= \frac{3\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$

$= \frac{3}{x+2}$

2. $\frac{1}{5x^2y^3} - \frac{8}{25xy^2}$ Get a common denominator. LCD= $25x^2y^3$

$= \frac{5}{5} \cdot \frac{1}{5x^2y^3} - \frac{8}{25xy^2} \cdot \frac{xy}{xy}$

$= \frac{5}{25x^2y^3} - \frac{8xy}{25x^2y^3}$

$= \frac{5-8xy}{25x^2y^3}$ Since it is not possible to factor the numerator, it is not possible to simplify further.

3. $\frac{x}{x-1} + \frac{x+7}{x^2-1} - \frac{x-2}{x+1}$ Factor all denominators, if possible.

$= \frac{x}{x-1} + \frac{x+7}{(x+1)(x-1)} - \frac{x-2}{x+1}$ Get a common denominator. LCD= $(x-1)(x+1)$

$$= \frac{x+1}{x+1} \cdot \frac{x}{x-1} + \frac{x+7}{(x+1)(x-1)} - \frac{x-2}{x+1} \cdot \frac{x-1}{x-1}$$

$$= \frac{x(x+1)}{(x+1)(x-1)} + \frac{(x+7)(x-1)}{(x+1)(x-1)} - \frac{(x-2)(x-1)}{(x+1)(x-1)}$$ Multiply numerators.

$$= \frac{x^2+x}{(x+1)(x-1)} + \frac{x+7}{(x+1)(x-1)} - \frac{x^2-3x+2}{(x+1)(x-1)}$$ Distribute negative.

$$= \frac{x^2+x}{(x+1)(x-1)} + \frac{x+7}{(x+1)(x-1)} + \frac{-(x^2-3x+2)}{(x+1)(x-1)}$$

$$= \frac{\cancel{x^2}+x+x+7\cancel{-x^2}+3x-2}{(x+1)(x-1)}$$ Combine like terms in the numerator.

$$= \frac{5x+5}{(x+1)(x-1)}$$ Simplify (factor numerator and cancel, if possible).

$$= \frac{5(\cancel{x+1})}{(\cancel{x+1})(x-1)}$$

$$= \frac{5}{x-1}$$

4. $$\frac{1}{x+y} - \frac{1}{x-y} - \frac{2y}{y^2-x^2}$$

$$= \frac{1}{x+y} - \frac{1}{x-y} - \frac{2y}{(y+x)(y-x)}$$ Factor out -1 from $(y-x)$ to get $-(x-y)$. Move negative to the front of the fraction.

$$= \frac{1}{x+y} - \frac{1}{x-y} - -\frac{2y}{(y+x)(x-y)}$$ The double negative becomes a + and we can rewrite $(y+x)$ as $(x+y)$.

$$= \frac{1}{x+y} - \frac{1}{x-y} + \frac{2y}{(x+y)(x-y)}$$ Get a common denominator.

LCD = $(x+y)(x-y)$

$$=\frac{x-y}{x-y}\cdot\frac{1}{x+y}-\frac{1}{x-y}\cdot\frac{x+y}{x+y}+\frac{2y}{(x+y)(x-y)}$$

$$=\frac{1(x-y)}{(x+y)(x-y)}-\frac{1(x+y)}{(x+y)(x-y)}+\frac{2y}{(x+y)(x-y)}$$ Distribute negative.

$$=\frac{x-y}{(x+y)(x-y)}+\frac{-x-y}{(x+y)(x-y)}+\frac{2y}{(x+y)(x-y)}$$

$$=\frac{x-y-x-y+2y}{(x+y)(x-y)}$$

$$=\frac{0}{(x+y)(x-y)}$$

$$=0$$

5. $\frac{x+8}{x-3}-\frac{x-14}{3-x}$ Factor out -1 from $3-x$.

$$=\frac{x+8}{x-3}-\frac{x-14}{-(x-3)}$$ Move negative in front of fraction.

$$=\frac{x+8}{x-3}--\frac{x-14}{x-3}$$ Rewrite double negative as +.

$$=\frac{x+8}{x-3}+\frac{x-14}{x-3}$$

$$=\frac{x+8+x-14}{x-3}$$

$$=\frac{2x-6}{x-3}$$ Simplify

$$=\frac{2\cancel{(x-3)}}{\cancel{x-3}}$$

=2

6. $2x+\frac{3x}{5x-2}$

$=\frac{2x}{1}+\frac{3x}{5x-2}$

$=\frac{5x-2}{5x-2}\cdot\frac{2x}{1}+\frac{3x}{5x-2}$

$=\frac{10x^2-4x}{5x-2}+\frac{3x}{5x-2}$

$=\frac{10x^2-4x+3x}{5x-2}$

$=\frac{10x^2-x}{5x-2}$

$=\frac{x(10x-1)}{5x-2}$

7. $\frac{2x}{x^2-4}-\frac{1}{(x-2)(x-1)}+\frac{x+1}{(x+2)(x-1)}$ Factor denominators, if possible.

$=\frac{2x}{(x+2)(x-2)}-\frac{1}{(x-2)(x-1)}+\frac{x+1}{(x+2)(x-1)}$ LCD= $(x+2)(x-2)(x-1)$

$=\frac{x-1}{x-1}\cdot\frac{2x}{(x+2)(x-2)}-\frac{1}{(x-2)(x-1)}\cdot\frac{x+2}{x+2}+\frac{x+1}{(x+2)(x-1)}\cdot\frac{x-2}{x-2}$

$=\frac{2x(x-1)}{(x+2)(x-2)(x-1)}-\frac{1(x+2)}{(x-2)(x-1)(x+2)}+\frac{(x+1)(x-2)}{(x+2)(x-1)}$ Multiply

$=\frac{2x^2-2x}{(x+2)(x-2)(x-1)}+\frac{-(x+2)}{(x+2)(x-2)(x-1)}+\frac{x^2-x-2}{(x+2)(x-2)(x-1)}$

$=\frac{2x^2-2x-x-2+x^2-x-2}{(x+2)(x-2)(x-1)}$

$$= \frac{3x^2-4x-4}{(x+2)(x-2)(x-1)}$$

$$= \frac{(3x+2)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}(x-1)}$$

$$= \frac{3x+2}{(x+2)(x-1)} \text{ or } \frac{3x+2}{x^2+x-2}$$

8. $\frac{3}{x}\left(\frac{7}{x+3} - \frac{5}{x-3}\right)$ Use Order of Operations – Parentheses First

Parentheses
Exponents
Multiplication/Division
Addition/Subtraction

$$= \frac{3}{x}\left(\frac{x-3}{x-3} \cdot \frac{7}{x+3} - \frac{5}{x-3} \cdot \frac{x+3}{x+3}\right)$$ LCD= $(x+3)(x-3)$

$$= \frac{3}{x}\left(\frac{7(x-3)}{(x+3)(x-3)} - \frac{5(x+3)}{(x-3)(x+3)}\right)$$

$$= \frac{3}{x}\left(\frac{7x-21}{(x+3)(x-3)} - \frac{5x+15}{(x+3)(x-3)}\right)$$ Distribute negative to numerator.

$$= \frac{3}{x}\left(\frac{7x-21}{(x+3)(x-3)} + \frac{-(5x+15)}{(x+3)(x-3)}\right)$$

$$= \frac{3}{x}\left(\frac{7x-21-5x-15}{(x+3)(x-3)}\right)$$

$$= \frac{3}{x}\left(\frac{2x-36}{(x+3)(x-3)}\right)$$ Multiply.

$$= \frac{3(2x-36)}{x(x+3)(x-3)}$$

$$= \frac{6x-108}{x(x+3)(x-3)} \text{ or } \frac{6(x-18)}{x(x+3)(x-3)}$$

9. $\left(\frac{3}{5}+\frac{1}{x}\right)\left(\frac{9}{x}-\frac{1}{4}\right)$ Use order of operations and work inside of each parentheses to get a common denominator.

$$=\left(\frac{x}{x}\cdot\frac{3}{5}+\frac{1}{x}\cdot\frac{5}{5}\right)\left(\frac{4}{4}\cdot\frac{9}{x}-\frac{1}{4}\cdot\frac{x}{x}\right)$$

$$=\left(\frac{3x}{5x}+\frac{5}{5x}\right)\left(\frac{36}{4x}-\frac{x}{4x}\right)$$

$$=\left(\frac{3x+5}{5x}\right)\left(\frac{36-x}{4x}\right)$$

$=\frac{(3x+5)(36-x)}{5x(4x)}$ FOIL the numerator and multiply denominator.

$=\frac{108x-3x^2+180-5x}{20x^2}$ Simplify numerator and write it in descending order.

$$=\frac{-3x^2+103x+180}{20x^2}$$

The next example has negative exponents, but it can be written as a sum of rational expressions.

10. $y^{-1}+(5y)^{-2}$ Rewrite without negative exponents.

$=\frac{1}{y}+\left(\frac{1}{5y}\right)^2$ Use Power of a Quotient rule to rewrite without parentheses.

$=\frac{1}{y}+\frac{1^2}{(5y)^2}$ Use Power of a Product rule to rewrite without parentheses.

$$= \frac{1}{y} + \frac{1}{25y^2}$$ Get a common denominator. LCD= $25y^2$

$$= \frac{25y}{25y} \cdot \frac{1}{y} + \frac{1}{25y^2}$$

$$= \frac{25y}{25y^2} + \frac{1}{25y^2}$$

$$= \frac{25y+1}{25y^2}$$