1.8 Rational Expressions: Simplifying, Multiplying and Dividing

If we know how to work with fractions, then the procedures we use when working with rational expressions will not be new. Here, we will briefly review the processes with numbers prior to doing the algebra with rational expressions to remind our students of the connections between arithmetic and algebra. We will begin with simplifying fractions and then rational expressions, but first we need to know what a rational expression is:

> **A <u>rational expression</u> is any expression that can be written as a ratio of polynomials in fraction form.**

For example, $\frac{x+3}{x^2-5x+1}$ is a rational expression because it is a ratio of polynomials written in fraction form. Recall that numbers are also polynomials (of degree 0), so we could say that the rational number $\frac{21}{15}$, for example, is also a rational expression since it is a ratio of polynomials. So we are really just generalizing the processes that we use on these rational numbers to rational expressions.

I. Simplifying Rational Expressions

Recall in arithmetic, when we reduce (simplify), we factor and cancel.

$\frac{21}{15} = \frac{7\cdot\cancel{3}}{5\cdot\cancel{3}} = \frac{7}{5}\cdot\frac{3}{3} = \frac{7}{5}$

We can cancel the 3's since they are factors (factors are things being multiplied together).

$\frac{3+7}{3+5} = \frac{10}{8} = \frac{\cancel{2}\cdot5}{\cancel{2}\cdot4} = \frac{5}{4}$

We must add and then simplify!

Be careful, we can't cancel unless we have factors (that means multiplication between them). Here, if we were to try to cancel the 3's, we would not get the right answer, would we?

As seen in the previous example, we cannot cancel over addition. We must first add and then simplify, if possible. So, if we have $\frac{x+3}{x+9}$,

we cannot cancel since the terms are being added and there are no like terms to combine. Thus, it is not possible to simplify.

But, if we have $\frac{3x}{9x} = \frac{\cancel{3 \cdot x}}{3 \cdot \cancel{3 \cdot x}} = \frac{1}{3}$, we are able to cancel 3 and *x* since they are factors. Since the numerator does not have any factors remaining, we must use a 1 as a placeholder in the numerator.

What about binomial factors?: $\frac{\cancel{(x+2)}(x-1)}{\cancel{(x+2)}(x+3)} = \frac{x-1}{x+3}$

As you can see, we can cancel any *factors* that "match" regardless of the number of terms in the factor.

Examples

Recall from section 1.2, we can subtract exponents precisely because we can cancel factors :

1. $\frac{9m^6n^5}{6m^4n^2} = \frac{\cancel{3} \cdot 3 \cdot \cancel{m \cdot m \cdot m \cdot m} \cdot m \cdot m \cdot \cancel{n \cdot n} \cdot n \cdot n \cdot n}{\cancel{3} \cdot 2 \cdot \cancel{m \cdot m \cdot m \cdot m \cdot n \cdot n}} = \frac{3 \cdot m \cdot m \cdot n \cdot n \cdot n}{2} = \frac{3m^2n^3}{2}$

If we rewrite the numerator and the denominator as factors, we can see the "matching" factors that may be cancelled. Of course, this is the long way to do this problem! It is much easier to use the exponent rules here from section 1.2.

2. $\frac{x^2+4x+4}{x^2+5x+6} = \frac{(x+2)\cancel{(x+2)}}{(x+3)\cancel{(x+2)}}$

$= \frac{x+2}{x+3}$

Consider the given rational expression. Since all of the terms are being "added" together, we cannot cancel any terms. But, we can rewrite the numerator and the denominator in factored form so that we can cancel "matching" factors. Whatever does not cancel is part of the answer!

3. $\frac{y+x}{x^2-y^2} = \frac{y+x}{(x+y)(x-y)}$ Factor the numerator and the denominator and rewrite $y + x$ as $x + y$.

$= \frac{\cancel{x+y}}{\cancel{(x+y)}(x-y)}$ Cancel the "matching" factors.

$= \frac{1}{x-y}$ Make sure to place a 1 in the numerator when writing out the answer.

4. $\frac{12-3x^2}{x^2-x-2} = \frac{-3x^2+12}{x^2-x-2}$ Rewrite numerator in descending order.

$= \frac{-3(x^2-4)}{(x-2)(x+1)}$ Factor numerator and denominator. Make sure to continue factoring until everything is factored completely.

$= \frac{-3(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x+1)}$ Cancel "matching" factors.

$= \frac{-3(x+2)}{x+1}$ or $-\frac{3(x+2)}{x+1}$ Write out answer using the remaining factors.

Alternate method:

$$\frac{12-3x^2}{x^2-x-2} = \frac{3(4-x^2)}{(x-2)(x+1)}$$

Factor numerator and denominator.

Make sure to continue factoring until everything is factored completely.

$$= \frac{3(2+x)(2-x)}{(x-2)(x+1)}$$

Rewrite $2+x$ as $x+2$ and then factor out a -1 from $(2-x)$ which gives $-1(x-2)$. We will move the -1 to the front of the numerator.

$$= \frac{-1 \cdot 3(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x+1)}$$

Cancel "matching" factors.

$$= \frac{-3(x+2)}{x+1} \text{ or } -\frac{3(x+2)}{x+1}$$

Write out answer using the remaining factors.

5. $$\frac{x^2+2x-15}{25-x^2} = \frac{(x+5)(x-3)}{-(x^2-25)}$$

Factor the numerator and the denominator. Make sure to factor completely.

$$= -\frac{\cancel{(x+5)}(x-3)}{\cancel{(x+5)}(x-5)}$$

Cancel "matching" factors.

$$= -\frac{x-3}{x-5} \text{ or } \frac{3-x}{x-5} \text{ or } \frac{x-3}{5-x}$$

Write answer using remaining factors.

Note: When a fraction is preceded by a negative sign, we can move that negative sign to the numerator (and then distribute) or we can move that negative sign to the denominator (and then distribute).

6. $\frac{x^3+8}{x^2-2x+4} = \frac{(x+2)(x^2-2x+4)}{x^2-2x+4}$ Factor. You must use the Sum of Cubes Formula to factor the numerator. The denominator is Prime (not factorable).

$= \frac{(x+2)\cancel{(x^2-2x+4)}}{\cancel{x^2-2x+4}}$ Cancel "matching" factors.

$= x + 2$ Write the answer using the remaining factors. Since the denominator does not have any remaining factors, a 1 is the placeholder. But a 1 in any denominator simplifies to a whole number that is the same as the numerator.

7. $\frac{ax+by+ay+bx}{a^2-b^2}$ It may be easiest to factor the numerator separately since it must be factored by grouping.

$$\begin{aligned} ax + by + ay + bx &= ax + bx + ay + by \\ &= x(a + b) + y(a + b) \\ &= (a + b)(x + y) \end{aligned}$$

$\frac{ax+by+ay+bx}{a^2-b^2} = \frac{(a+b)(x+y)}{(a+b)(a-b)}$ Factor completely.

$= \frac{\cancel{(a+b)}(x+y)}{\cancel{(a+b)}(a-b)}$ Cancel "matching" factors.

$= \frac{x+y}{a-b}$ Write answer using the remaining factors.

II. <u>Multiplying and Dividing Rational Expressions</u>

Arithmetic: $\frac{3}{8} \cdot \frac{6}{15} = \frac{3}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 3}{3 \cdot 5} = \frac{\cancel{3} \cdot \cancel{2} \cdot 3}{\cancel{2} \cdot 2 \cdot 2 \cdot \cancel{3} \cdot 5} = \frac{3}{2 \cdot 2 \cdot 5} = \frac{3}{20}$

With rational expressions, we do the same process (factor and cancel):

$$\frac{x^2+4x+4}{x^2-9} \cdot \frac{x^2+6x+9}{x^2+7x+10} = \frac{\cancel{(x+2)}(x+2)\cancel{(x+3)}(x+3)}{\cancel{(x+3)}(x-3)\cancel{(x+2)}(x+5)} = \frac{(x+2)(x+3)}{(x-3)(x+5)} \quad \text{or} \quad \frac{x^2+5x+6}{x^2+2x-15}$$

factored form not factored form

<u>Examples: Multiplication</u>

1. $$(x+2) \cdot \left(\frac{9x+4}{x^2-4}\right) = \frac{x+2}{1} \cdot \frac{9x+4}{x^2-4}$$

$$= \frac{\cancel{(x+2)}(9x+4)}{\cancel{(x+2)}(x-2)}$$

$$= \frac{9x+4}{x-2}$$

2. $$\frac{3t^2-t-2}{6t^2-5t-6} \cdot \frac{4t^2-9}{2t^2+5t+3} = \frac{\cancel{(3t+2)}(t-1)\cancel{(2t+3)}\cancel{(2t-3)}}{\cancel{(2t-3)}\cancel{(3t+2)}\cancel{(2t+3)}(t+1)}$$

$$= \frac{t-1}{t+1}$$

Examples: Division

Arithmetic example: $\frac{4}{9} \div \frac{8}{15} = \frac{4}{9} \cdot \frac{15}{8} = \frac{\cancel{2\cdot2}}{\cancel{3}\cdot3} \cdot \frac{\cancel{3}\cdot5}{\cancel{2\cdot2}\cdot2} = \frac{5}{3\cdot2} = \frac{5}{6}$

3. $$\frac{8y^2-14y-15}{6y^2-11y-10} \div \frac{3y^2-7y-6}{4y^2-9y-9} = \frac{8y^2-14y-15}{6y^2-11y-10} \cdot \frac{4y^2-9y-9}{8y^2-14y-15}$$

$$= \frac{(\cancel{2y-5})(4y+3)(4y+3)(\cancel{y-3})}{(\cancel{2y-5})(3y+2)(3y+2)(\cancel{y-3})}$$

$$= \frac{(4y+3)^2}{(3y+2)^2} \text{ or } \left(\frac{4y+3}{3y+2}\right)^2$$

4. $$\frac{2x^2-2x-4}{x^2+2x-8} \cdot \frac{3x^2+15x}{x+1} \div \frac{4x^2-100}{x^2-x-20} = \frac{2x^2-2x-4}{x^2+2x-8} \cdot \frac{3x^2+15x}{x+1} \cdot \frac{x^2-x-20}{4x^2-100}$$

$$= \frac{2\left(x^2-x-2\right)}{x^2+2x-8} \cdot \frac{3x(x+5)}{x+1} \cdot \frac{x^2-x-20}{4(x^2-25)}$$

$$= \frac{\cancel{2}(\cancel{x-2})(\cancel{x+1})\cdot3x(\cancel{x+5})(\cancel{x+4})(\cancel{x-5})}{(\cancel{x+4})(\cancel{x-2})(\cancel{x+1})\cdot\cancel{2}\cdot2(\cancel{x+5})(\cancel{x-5})}$$

$$= \frac{3x}{2}$$