1.7 Solving Equations by Factoring

Now, we are back to solving equations! We had to learn how to factor in order to proceed with polynomial equations because factoring is part of the solving process for these types of equations. (This is just one of the reasons factoring is a very valuable skill - you will also use it later to solve inequalities in this course as well as to graph polynomial functions in your next course and beyond.)

In order to understand the technique used for solving polynomial equations, we must first recall one of the basic algebraic properties:

Zero Product Property: $\boldsymbol{a \cdot b = 0}$ **if and only if** $\boldsymbol{a = 0}$ **or** $\boldsymbol{b = 0}$**.**

We know this is true for numbers. We cannot have two numbers that multiply to be 0 unless at least one of those numbers is 0. Conversely, if either number is 0, then the product must be 0. We can apply this when we have polynomials as well because the variables represent numbers, which means the factors represent numbers as well, and so the same logic applies.

For example, if $(x+2)(x-3)=0$, then we can think of $x+2$ as our $'a'$ and $x-3$ as our $'b'$ in the Zero Product Property above. (They are both numbers since x represents a number, so it makes sense to think of it this way).

This means that if either $x+2$ is 0 or $x-3$ is 0, then the product of the two must be 0. To find out what values of x will make one or the other equal 0 is easy – just set each factor equal to 0. When the factors are linear, this is a very simple process!

$$\begin{array}{rcl} x+2 &=& 0 \\ -2 && -2 \\ \hline x &=& -2 \end{array} \qquad \begin{array}{rcl} x-3 &=& 0 \\ +3 && +3 \\ \hline x &=& 3 \end{array}$$

Now, check each solution to make sure the answer is correct. If you factored correctly, then they should both work!

Check:

For $x = -2$:
$(-2+2)(-2-3)$
$= 0 \cdot (-3)$
$= 0$

For $x = 3$
$(3+2)(3-3)$
$= 5 \cdot 0$
$= 0$

Now, we can solve polynomial equations by factoring the polynomial and setting each factor equal to 0. **Please make sure that you have 0 on one side of the equation before you factor.** This logic only makes sense when the product is 0. For example, just because two numbers multiply to be 15 does not mean you can set each of them equal to 15, does it? We know that $3 \cdot 5 = 15$, yet neither 3 nor 5 is equal to 15! But if two numbers multiply to be 0, we know that at least one of them is 0. **The only number we can say this about is 0.**

Examples

1. $x^2 + 5x + 4 = 0$ — Factor

$(x+1)(x+4) = 0$ — Set each factor equal to 0, then solve.

$$\begin{array}{rr} x+1=0 & x+4=0 \\ -1\quad -1 & -4\quad -4 \\ \hline x=-1 & x=-4 \end{array}$$

Make sure you check!

2. $2y^2 - 5y = -2$

$$\begin{array}{r} 2y^2-5y \quad = -2 \\ +2 \quad +2 \\ \hline 2y^2-5y+2=0 \end{array}$$

We need to get 0 on one side so we can factor and then use the zero product property to solve.

$(2y-1)(y-2) = 0$ — Factor.

$$\begin{array}{ll} 2y - 1 = 0 & y - 2 = 0 \\ \underline{\quad +1 \quad +1} & \underline{\quad +2 \quad +2} \\ 2y = 1 & y = 2 \\ \frac{2y}{2} = \frac{1}{2} & \\ y = \frac{1}{2} & \end{array}$$

Set each factor equal to 0, then solve.

Make sure you check!

So, $y = \frac{1}{2}$ and $y = 2$

3. $$\begin{array}{l} x^2 - 2x = -1 \\ \underline{\qquad\quad +1 \quad +1} \\ x^2 - 2x + 1 = 0 \end{array}$$

We need to get 0 on one side so we can factor and then use the zero product property to solve.

Factor

$(x - 1)(x - 1) = 0$

Set one factor equal to 0, then solve.

$$\begin{array}{l} x - 1 = 0 \\ \underline{\quad +1 \quad +1} \\ x = 1 \end{array}$$

(Since the factors are the same, use one.)

Make sure you check!

4. $$\begin{array}{l} 4x^2 = -8\text{x} \\ \underline{\quad +8\text{x} \quad +8\text{x}} \\ 4x^2 + 8x = 0 \end{array}$$

Get 0 on one side so we can factor and use the zero product property.
Factor.

$4x(x + 2) = 0$

Set each factor equal to 0, then solve.

$$\begin{array}{ll} 4x = 0 & x + 2 = 0 \\ \frac{4x}{4} = \frac{0}{4} & \underline{\quad -2 \quad -2} \\ x = 0 & x = -2 \end{array}$$

Make sure you check!

Note: Don't divide both sides by a variable because you will lose some of your solutions. Consider the previous example, done incorrectly below to show what happens:

$$\frac{4x^2}{4x} = \frac{-8x}{4x}$$

$$x = -2$$

You can see that the solution $x = 0$ was lost! So, dividing both sides of an equation by a variable will cause us to lose possible solutions.

5. $x^2 = \frac{1}{2}(x+1)$

To get 0 on one side, clear the fraction first by multiplying both sides by 2.

$$2 \cdot x^2 = 2 \cdot \frac{1}{2}(x+1)$$

$$\begin{array}{rcl} 2x^2 & = & x+1 \\ -x-1 & & -x-1 \\ \hline 2x^2 - x - 1 & = & 0 \end{array}$$

Now, "move" everything to one side.

Factor.

$$(2x+1)(x-1) = 0$$

Set each factor equal to 0, then solve.

$$\begin{array}{rcl} 2x + 1 & = & 0 \\ -1 & & -1 \\ \hline 2x & = & -1 \end{array} \qquad \begin{array}{rcl} x - 1 & = & 0 \\ -1 & & -1 \\ \hline x & = & -1 \end{array}$$

$$\frac{2x}{2} = \frac{-1}{2}$$

$$x = \frac{-1}{2}$$

So, $x = \frac{-1}{2}$ and $x = -1$

6. $x^3 + 4x^2 = 25(x+4)$ Distribute the 25 and then get 0 on one side so you can factor.

$$\begin{array}{l} x^3 + 4x^2 \qquad\qquad = 25x + 100 \\ \qquad -25x - 100 \quad -25x - 100 \\ \hline \underbrace{x^3 + 4x^2} \quad \underbrace{-25x - 100} = 0 \end{array}$$

Now, Factor by grouping.

$(x^3 + 4x^2) + (-25x - 100) = 0$

$x^2(x+4) \; - 25(x+4) = 0$

$(x+4)(x^2 - 25) = 0$ Keep factoring!

$(x+4)(x+5)(x-5) = 0$ Set each factor equal to 0 and then solve.

$$\begin{array}{c} x + 4 = 0 \\ -4 \quad -4 \\ \hline x = -4 \end{array} \qquad \begin{array}{c} x + 5 = 0 \\ -5 \quad -5 \\ \hline x = -5 \end{array} \qquad \begin{array}{c} x - 5 = 0 \\ +5 \quad +5 \\ \hline x = 5 \end{array}$$

So, $x = -4$ and $x = -5$ and $x = 5$.

7. $9(y+4) = y^3 + 4y^2$ Distribute the 9 and then get 0 on one side so you can factor.

$$\begin{array}{l} 9y + 36 \qquad\qquad = y^3 + 4y^2 \\ -9y - 36 \qquad\qquad\qquad\qquad - 9y - 36 \\ \hline \qquad 0 \qquad\qquad = \underbrace{y^3 + 4y^2} \quad \underbrace{-9y - 36} \end{array}$$

Factor by grouping.

$(y^3 + 4y^2) + (-9y - 36) = 0$

$y^2(y+4) \; - 9(y+4) \quad = 0$

$(y+4)(y^2-9)=0$ Keep factoring!

$(y+4)(y+3)(y-3)=0$ Set each factor equal to 0 and solve.

$$\begin{array}{ccc} y+4=0 & y+3=0 & y-3=0 \\ \underline{\;-4\quad -4\;} & \underline{\;-3\quad -3\;} & \underline{\;+3\quad +3\;} \\ y=-4 & y=-3 & y=3 \end{array}$$

So, $y=-4$ and $y=-3$ and $y=3$.

8. $\frac{x^2(6x+37)}{35}=x$ Multiply by 35 to clear the fraction.

$35\cdot\frac{x^2(6x+37)}{35}=x\cdot 35$ You can cancel the 35's on the lhs.

$x^2(6x+37)=35x$ Distribute on the left hand side.

$$\begin{array}{rl} 6x^3+37x^2 \qquad\quad & =35x \\ \underline{\qquad\qquad -35x} & \underline{\;-35x} \\ 6x^3+37x^2-35x & =0 \end{array}$$

Get 0 on one side so you can factor.

Factor out the GCF.

$x(6x^2+37x-35)=0$ Factor the trinomial.

$x(x+7)(6x-5)=0$ Set each factor equal to 0 and solve.

$$\begin{array}{ccc} x=0 & x+7=0 & 6x-5=0 \\ & \underline{\;-7\quad -7\;} & \underline{\;+5\quad +5\;} \\ & x=-7 & 6x=5 \\ & & \frac{6x}{6}=\frac{5}{6} \\ & & x=\frac{5}{6} \end{array}$$

So, $x=0$ and $x=-7$ and $x=\frac{5}{6}$.