1.6 Factoring Binomials

Binomials are polynomials that have two terms. In order to factor them, we need to think a bit backwards about it. Consider what happened when we multiplied together two binomials that were the same except for the sign between the terms, i.e. of the form $(a+b)(a-b)$. Recall that the middle terms dropped out and we obtained a^2-b^2. If we thought about this in the reverse, we would realize that a^2-b^2 factors into $(a+b)(a-b)$.

I. Difference of Squares Formula: $\boldsymbol{a^2-b^2=(a+b)(a-b)}$

Why? $(a+b)(a-b)=a^2\underbrace{-ab+ab}_{0}-b^2$

$=a^2-b^2$

<u>Examples</u> Factor the given binomial.

1. $4x^2-9$

$=(2x)^2-3^2$

Let $a=2x$ and $b=3$

$4x^2-9=(2x+3)(2x-3)$

2. x^4-16

$=(x^2)^2-4^2$

Let $a=x^2$ and $b=4$

$x^4-16=(x^2+4)\underbrace{(x^2-4)}_{a=x \text{ and } b=2.}$ Now, factor again!

$x^4-16=(x^2+4)\underbrace{(x^2-4)}$

$=(x^2+4)(x+2)(x-2)$

3. $16x^2 - 49y^2$

$= (4x)^2 - (7y)^2$

Let $a = 4x$ and $b = 7y$

$16x^2 - 49y^2 = (4x + 7y)(4x - 7y)$

4. $8x^3 - 18x$

$= 2x(4x^2 - 9)$ Factor out GCF first (always!)

$= 2x[(2x)^2 - (3)^2]$

Let $a = 2x$ and $b = 3$

$= 2x(2x + 3)(2x - 3)$

5. $4x^2 + 9$

This binomial is prime (since it is a sum of squares "+" not a difference of squares "-").

While this binomial does not factor in the real numbers, you will see later that if we allow "imaginary numbers", then we can factor it using those by rewriting it as a difference of squares: $4x^2 - (-9)$.

6. $75x^4 - 12x^2$

$= 3x^2(25x^2 - 4)$ Factor out GCF first

$= 3x^2[(5x)^2 - 2^2]$

Let $a = 5x$ and $b = 2$

$= 3x^2(5x + 2)(5x - 2)$

7. $50x^4y^2 - 98z^6$

$= 2\left(\underbrace{25x^4y^2 - 49z^6}\right)$ Factor out GCF first.

$= 2[(5x^2y)^2 - (7z^3)^2]$ Let $a = 5x^2y$ and $b = 7z^3$

$= 2(5x^2y + 7z^3)(5x^2y - 7z^3)$

8. $a^8 - b^8$

$$= [(a^4)^2 - (b^4)^2]$$
$$= (a^4 + b^4)\,(a^4 - b^4)$$
$$= (a^4 + b^4)(a^2 + b^2)(a^2 - b^2)$$
$$= (a^4 + b^4)\ \ (a^2 + b^2)(a + b)(a - b)$$

In this example, we need to break down the difference of squares multiple times. Note that:
$a^4 - b^4 = (a^2)^2 - (b^2)^2$
$= (a^2 + b^2)(a^2 - b^2)$
Also note that:
$a^2 - b^2 = (a + b)(a - b)$

Notice that we must keep factoring as long as we can, just like we do with numbers. For example, when you factor the number 30, you don't stop at $3 \cdot 10$, but rather you also break down the 10 to get $30 = 3 \cdot 2 \cdot 5$. You factor until all of your factors are prime. The same thing is true in algebra.

Next, we look at some trickier examples where we need to group terms in different ways to obtain a difference of two squares.

<u>(Trickier) Examples</u>

9. $(y + 5)^2 - 16$
Let $a = (y + 5)$ and $b = 4$,

Then

$$(y + 5)^2 - 16 = (y + 5 + 4)(y + 5 - 4)$$
$$= (y + 9)(y + 1)$$

10. $\underbrace{x^2 + 2xy + y^2} - z^4$ Factor the perfect square trinomial.

$= (x + y)^2 \quad - z^4$ Let $a = (x + y)$ and $b = z^2$

$= (x + y + z^2)(x + y - z^2)$

Next, we see that we can use the difference of squares to get some expressions in a form that will allow for factoring by grouping.

11. $\underbrace{x^2 - y^2} + \underbrace{6x + 6y}$ Factor difference of squares from the first group, then factor the GCF from the second group.

$= \underbrace{(x+y)(x-y)} + \underbrace{6(x+y)}$ Now, factor out GCF $(x+y)$
$= (x+y)(x-y+6)$

12. $3m^2 - 3n^2 + 12m + 12n$ Factor out GCF
$= 3\,\underbrace{(m^2 - n^2 + 4m + 4n)}$ Factor by Grouping
$= 3\left[\underbrace{(m^2 - n^2)} + \underbrace{(4m + 4n)}\right]$ Factor difference of squares & GCF
$= 3[(m+n)(m-n) + 4(m+n)]$ Factor out GCF $(m+n)$
$= 3(m+n)(m-n+4)$

Now that we have explored the difference of squares, we can look at two more formulas that are not as straightforward to memorize, but easy to show that they work. Just multiply out the right hand side to see that these formulas are true:

II. Difference of Cubes Formula: $\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

Sum of Cubes Formula: $\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$

Once you recognize your binomial as either a sum or a difference of cubes, figure out what a and b are by thinking of what you need to cube to obtain each term and then just plug them into the formula.

Examples

1. $8x^3 - 27$
$= (2x)^3 - 3^3$
Let $a = 2x$ and $b = 3$

Now, plug these into the difference of cubes formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

You should get

$$8x^3 - 27 = (2x)^3 - 3^3 = (2x - 3)(4x^2 + 6x + 9)$$

2. $64x^3 + 1$
$= (4x)^3 + 1^3$
Let $a = 4x$ and $b = 1$

Now, plug these into the sum of cubes formula:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

You should get

$64x^3 + 1 == (4x)^3 + 1^3 = (4x + 1)(16x^2 - 4x + 1)$

3. $y^3 + 343$
Let $a = y$ and $b = 7$

$y^3 + 343 = (y + 7)(y^2 - 7y + 49)$

4. $27x^3 - \frac{1}{8}y^3$

Let $a = 3x$ and $b = \frac{1}{2}y$

$$27x^3 - \frac{1}{8}y^3 = \left(3x - \frac{1}{2}y\right)\left(9x^2 + \frac{3}{2}xy + \frac{1}{4}y^2\right)$$

5. $16m^3n^6 + 54p^9$ Factor out GCF.

$$16m^3n^6 + 54p^9 = 2\,\underbrace{(8m^3n^6 + 27p^9)}_{a\,=\,2mn^2 \text{ and } b\,=\,3p^3}$$

$$\begin{aligned} 16m^3n^6 + 54p^9 &= 2\,\underbrace{(8m^3n^6 + 27p^9)} \\ &= 2(2mn^2 + 3p^3)(4m^2n^4 - 6mn^2p^3 + 9p^6) \end{aligned}$$

6. $3x^4y^3 - 24xz^6$ Factor out GCF.

$$3x^4y^3 - 24xz^6 = 3x\,\underbrace{(x^3y^3 - 8z^6)}_{a\,=\,xy \text{ and } b\,=\,2z^2}$$

$$\begin{aligned} 3x^4y^3 - 24xz^6 &= 3x\,\underbrace{(x^3y^3 - 8z^6)} \\ &= 3x(xy - 2z^2)(x^2y^2 + 2xyz^2 + 4z^4) \end{aligned}$$

III. Factoring Technique

The following is an algorithm that you could use to guide you through any factoring of polynomials that you encounter in this course. (You will add a little bit to this algorithm when you learn *The Rational Root Theorem* in a future course.)

1. Factor out GCF, if possible.

2. Count the number of terms for whatever is left after step 1.

   2 terms: Use a formula, if possible.
   -difference of squares, or
   -difference of cubes, or
   -sum of cubes

   3 terms: Use the Rainbow method or
   use the ac-method

   4 terms: Factor by grouping or it is similar to a "tricky" one
   as demonstrated in the notes.

3. Look at each factor to see if completely factored. If not, repeat step 2.