1.4 GCF and Factoring by Grouping

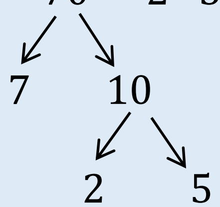
Now, we will begin learning about factoring. For the next three lessons, we will learn to factor certain types of polynomials, but basic to all factoring is pulling out any greatest common factor (GCF) first, so this is where we begin. We will first relate the concept of factoring to the idea of multiplication, and using numbers is a nice way to make that connection. Then we can apply the same idea to polynomials.

Factoring can be thought of as the "undoing" of multiplication, just as division can. Factoring is related to division because we really do divide when we factor. It's just that we write the factorization a little differently than the division.

Recall in arithmetic:

Multiplication $\quad 5 \cdot 2 \cdot 7 = 70$

Factoring $\quad 70 = 2 \cdot 5 \cdot 7$



Factoring involves breaking down the number into its prime factors by dividing the number repeatedly. Recall that this is called a "factor tree".

We can see that factoring goes backwards from multiplication, so we can think of it as "undoing" multiplication.

The same idea applies to polynomials:

Multiplication $\quad 5(2x + 1) = 10x + 5$

Factoring $\quad 10x + 5 \quad = 5(2x + 1)$

Most basic type of factoring is pulling out the greatest common factor (GCF), which is what we just did in the example above.

Examples

Factor each of the following polynomials by factoring out the GCF.

1. $x^2 + 3x = x(x + 3)$

   In this example, both terms have a factor of x, so we can factor that out. To find what is left inside, simply divide each term by x. You can distribute to check that your answer gives you what you started with.

2. $2x^3y^2 - 8xy + 4x^4y^3 = 2xy(x^2y - 4 + 2x^3y^2)$

   In this example, all three terms have a factor of $2xy$. To see this, first look at each number to see that they have a GCF of 2. Then look at the powers of x and take the smallest power of x. Then do the same thing for y. To find what is left inside, simply divide each term by $2xy$. You should distribute to check that your answer gives you what you started with.

3. $3x^2y - 27xy^2 = 3xy(x - 9y)$

   We pulled out the GCF here, but that was all we could do.

4. $\frac{3}{2}t^2y^4 - \frac{1}{2}ty^4 - \frac{5}{2}ry^3 = \frac{1}{2}y^3(3t^2y - ty - 5r)$

   When you have fractions involved, a great way to approach the number portion is to first consider the GCF of the numerator and then consider the GCF of the denominator. Here, the numerators have a GCF of 1 and the denominators have a GCF of 2. Not every term has t, so you cannot pull out any $t's$. The same thing applies to r.

The next example leads us to the concept of factoring by grouping....

5. $\underbrace{x^2(x+7)} + \underbrace{3(x+7)} = (x+7)(x^2+3)$

Here we have two terms with common factor of $(x+7)$. When we factor out $(x+7)$, we get the other factor from the remaining terms, (x^2+3) because if we divide each term by $x+7$, that is what remains.

If you are having trouble seeing this, consider the following instead:

$$x^2y + 3y$$

How would you factor this?

$$x^2y + 3y = y(x^2+3)$$

Now think about it.

In the original example, instead of y, we had $x+7$. What if you replace y with $x+7$ in the simpler example above?

Factoring by grouping

Factoring by grouping is typically used when we have an even number of terms ≥ 4. This is because we can then divide the terms in half and try to factor out the same thing from each half.

Consider the following example:

$$x^3 + 7x + 3x + 21$$

If we group the first two and the second two

$$\underbrace{x^3 + 7x^2} + \underbrace{3x + 21}$$

We can pull the GCF out of each one

$$\underbrace{x^2(x+7)} + \underbrace{3(x+7)}$$

Now, we see that each has the same thing leftover inside. This is crucial because we can only factor it out if it is the same factor for both.

$$= (x+7)(x^2+3)$$

All we did was add a layer in the beginning to the last example we did, and that layer was grouping.

Examples

Factor each of the following polynomials by grouping.

1. $\underbrace{2x+10} + \underbrace{3x^2+15x}$

$$= 2\underbrace{(x+5)} + 3x\underbrace{(x+5)}$$

$$= (x+5)(2+3x)$$

Note that there are many ways to write your answer since both addition and multiplication are commutative.

$$= (x+5)(3x+2)$$

$$= (3x+2)(x+5)$$

$$= (3x+2)(5+x)$$

etc…..

Sometimes you have choices about how to group if you rearrange the terms, but you will still arrive at the same answer.

2. $\underbrace{x+1}+\underbrace{y+xy}$

$= 1(x+1)+y(1+x)$

$= 1\,\underbrace{(x+1)}+y\,\underbrace{(x+1)}$

$= (x+1)(1+y)$ or $(x+1)(y+1)$ or $(y+1)(x+1)$ or …

3. $\underbrace{10x^4+5x^3y}\underbrace{-2x-y}$

$= 5x^3\,\underbrace{(2x+y)}-1\,\underbrace{(2x+y)}$

$= (2x+y)(5x^3-1)$

4. $12x^3y-4x^2y^3-6xy^2+2y^4$

$= 2y\left[\underbrace{6x^3-2x^2y^2}\underbrace{-3xy+y^3}\right]$

$2x^2\,\underbrace{(3x-y^2)}-y\,\underbrace{(3x-y^2)}$

$(3x-y^2)(2x^2-y)$

$= 2y\,(3x-y^2)(2x^2-y)$

Don't forget that it is always easier to pull out the GCF first.

Here, we are just working on the inside and then we will bring down and include the GCF at the end of the problem.

5. $3x^2y - 3y^2 - 3xy + 3xy^2$

$$= 3y\left[\underbrace{x^2 - y}\ \underbrace{-x + xy}\right]$$

Here, we had to rearrange the terms inside in order to get a common factor. If you group and what you left inside does not match, you will need to rearrange your terms and try again.

Note: if you get $a - b$ inside of one and $b - a$ inside of the other, simply pull out a negative since $b - a = -(a - b)$. We could have had this happen here if we rearranged the terms differently.

$$= 3y\left[\underbrace{x^2 - x}\ \underbrace{+xy - y}\right]$$

$$x\,\underbrace{(x-1)} + y\,\underbrace{(x-1)}$$

$$(x+y)(x-1)$$

$$= 3y\,(x+y)(x-1)$$