1.3 Operations with Polynomials

A *polynomial* is a sum (or difference) of real number multiples of whole number powers of a given variable (or variables). Said a little differently, if you add together whole number powers of a variable with a real number in front of each, you get a *polynomial*. Each piece of the polynomial that is being added is called a *term*. The number in front of each *term* is called a *coefficient*. Polynomials can have more than one variable in them.

<u>Examples</u>

Decide whether each expression is a polynomial or not.

1. $x^2 + 5x + 1$ Yes

This is a polynomial because all of the powers on the variables are whole numbers (there are no negative or fractional powers).

2. $3x^9y^2 - 8x^3y^5 + x - 7$ Yes

This is an example of a polynomial in two variables. Note that every power is a whole number here as well.

3. $\frac{3}{x} + x^5$ No

This is not a polynomial because there are negative exponents here since this can be written as $3x^{-1} + x^5$. The powers must ALL be whole numbers to be a polynomial.

4. $\sin x + \cos x - x^2 + \tan^2 x$ No

These are not even powers of a variable, but rather powers of “functions” of a variable (we will learn about functions later). If you take a course in trigonometry, you will learn about these particular functions of sine, cosine, and tangent.

5. e^x No

This is not a polynomial because the variable is in the exponent rather than the base. This is called an exponential and we will study them soon!

6. $\ln x$ No

This is not a polynomial, but rather a logarithm, and we read it as the "natural log of x". Here x is once again inside of a "function" or an operation. We will study these soon as well!

Note: Polynomials are the nicest kinds of expressions to work with and the best kinds of 'functions' to deal with due to their easy computation and graphing. It is easy to do calculus on them as well! Often in applications of higher mathematics, we try to approximate more complicated functions with polynomials for this reason, especially in engineering applications.

Terminology:

Polynomials are classified into different types (at least for the first few terms). This is done in order to distinguish between the kinds of polynomials when you are factoring them. Factoring will be important to help us solve polynomial equation. The meaning of each is given below.

$\underbrace{Poly}_{\text{many}}\ \underbrace{nomial}_{\text{terms}}$ (Greek origin)

monomial – one term
binomial – two terms
trinomial – three terms

The *degree* of a term is defined to be the sum of the powers of the term.

The *degree* of a polynomial is defined to be the largest of the degrees of the terms.

Often, we will write a polynomial in *descending order*. This means we put the terms in order from the highest degree to the lowest degree. We do this because it makes a polynomial easier to factor and we will need to factor in order to solve equations.

Examples

Give the degree of the polynomial and put it in descending order.

1. $$\underbrace{8x^5}_{5} + \underbrace{3x^3}_{3} - \underbrace{x^1}_{1} + \underbrace{1}_{0}$$

Degree of each term: 5 3 1 0

The degree of the polynomial is 5 (equal to largest degree of the terms).
Note: $1 = 1x^0$ (which means every constant has a degree of 0).

2. $$\underbrace{-2x^6}_{6} + \underbrace{x^2}_{2}$$

Degree of each term: 6 2

The degree of the polynomial is 6.

3. $$\underbrace{2x^3y}_{4} + \underbrace{7x^5y^2}_{7} - \underbrace{13x^8}_{8} + \underbrace{5}_{0}$$

Degree of each term: 4 7 8 0

The degree of a term is found by adding the exponents on the variables in one term.

The degree of the polynomial is 8.

Put in descending order: $-13x^8 + 7x^5y^2 + 2x^3y + 5$

4. $$\underbrace{-3x^2yz^9}_{12} + \underbrace{15xz}_{2} - \underbrace{6y^3z^5}_{8}$$

Degree of each term: 12 2 8

The degree of the polynomial is 12.

Put in descending order: $-3x^2yz^9 - 6y^3z^5 + 15xz$

Now that we know what a polynomial is, we can begin to work with polynomial expressions by adding, subtracting, and multiplying. We will also divide polynomials, but we will do that later in the course after we have worked a bit with rational expressions (the two are related). In order to add and subtract polynomials, we go back to the idea of combining "like" terms. We need to remember that adding (and subtracting) is really just counting. When we are counting things, those things need to be alike in order to put them in the same category (which is what counting does). If they are different, we cannot count them in the same category, and hence we cannot combine them. In general, combining "like" terms can be thought of as adding together terms that look exactly alike (except for possibly the number in front).

<u>Examples</u>

Simplify each expression completely.

1. $2\underbrace{x^2} + 3\underbrace{x^2} = 5x^2$

   Since both terms have identical variables and exponents, they are "like" terms, so we can count them in the same category. If we have 2 of these x^2's and then we have 3 more of these x^2's, then we have a total of 5 of them. We can just add the coefficients and keep the variable and exponent exactly the same. We could also think of this in terms of factoring out x^2:

$$2x^2 + 3x^2 = x^2(2+3) = x^2 \cdot 5 = 5x^2$$

2. $2x^2 + 3y^2$

These two terms cannot be combined since the variables are different! If we have 2 x^2's and y's, then we don't have 5 of the same thing, so we cannot combine them. Therefore, we would leave the answer in this form without any changes.

Even if we have things other than polynomials, we can apply this concept:

3. $2\underbrace{\sqrt{7}} + 3\underbrace{\sqrt{7}} = 5\sqrt{7}$

Since both are identical except for the numbers in front of $\sqrt{7}$, they are "like" terms and we add the coefficients and keep the rest exactly the same, even if we don't know what it means yet.

4. $2\underbrace{\cos x} + 3\underbrace{\cos x} = 5\cos x$

The same logic applies here.

5. $2\star + 3\star = 5\star$

This is a fun, silly example that shows two "like" terms because both terms have "stars", so here we are counting "stars". Just like the other examples, we add the coefficients and keep the "star".

6. $2\underbrace{x\sqrt{y}\sin z^3} + 3\underbrace{x\sqrt{y}\sin z^3} = 5x\sqrt{y}\sin z^3$

Even if we don't know what the expressions mean, we can still identify "like terms". These are "like" terms since both terms are exactly alike except for the coefficient (number in front). So, we add the coefficients and keep the rest exactly the same. Understanding this concept will help you in this course as well as in future classes!

Addition/Subtraction of Polynomials

When adding two polynomials, we combine "like" terms.

For example,

$$(3x^2 + 5x - 1) + (2x^2 - x + 4) = \underbrace{3x^2}\,\overline{\overline{+5x}}\,\overbrace{-1} + \underbrace{2x^2}\,\overline{\overline{-x}}\,\overbrace{+4}$$

$$= 5x^2 + 4x + 3$$

Note: We can "mark" the terms that are alike with the same thing so that they stand out visually.

To subtract two polynomials, we "distribute" the negative and then combine "like" terms.

For example,

$$(3x^2 + 5x - 1) - (2x^2 - x + 4) = \underbrace{3x^2}\,\overline{\overline{+5x}}\,\overbrace{-1}\,\underbrace{-2x^2}\,\overline{\overline{+x}}\,\overbrace{-4}$$

$$= x^2 + 6x - 5$$

Examples

Add or subtract each of the following polynomials and simplify the result completely.

3. $(3x^2y - 6xy + y^2 + 1) + (5xy^2 - 2xy + 3y^2 - x + 4)$

$$= 3x^2y \overbrace{-6xy} \overline{\overline{+y^2}} \underbrace{+1} + 5xy^2 \overbrace{-2xy} \overline{\overline{+3y^2}} - x \underbrace{+4}$$

$$= 3x^2y - 8xy + 4y^2 + 5 + 5xy^2 - x$$

In descending order: $3x^2y + 5xy^2 - 8xy + 4y^2 - x + 5$

4. $(7x^2yz^3 + 8xy - 4z^3 + 1) - (11xy + 8y^2 - 18x^2yz^3 + 3)$

$$= \underbrace{7x^2yz^3} \overbrace{+8xy} - 4z^3 \overline{\overline{+1}} \overbrace{-11xy} - 8y^2 \underbrace{+18x^2yz^3} \overline{\overline{-3}}$$

$$= 25x^2yz^3 - 3xy - 4z^2 - 8y^2 - 2$$

Multiplying Polynomials

Recall: $(2x^5)(3x^7) = 2 \cdot x^5 \cdot 3 \cdot x^7$ Remove parentheses

$= 2 \cdot 3 \cdot x^5 \cdot x^7$ Group numbers and like variables

$= 6x^{12}$ Multiply numbers and variables (by adding their exponents: $x^5 \cdot x^7 = x^{5+7} = x^{12}$)

Examples

Multiply each of the following polynomials and simplify the result completely.

1. $3x^2(2x^2 - 3x + 1)$

Distribute first:

$$= (3x^2)(2x^2) + (3x^2)(-3x) + (3x^2)(1)$$

And then multiply:

$$= 6x^4 - 9x^3 + 3x^2$$

2. $(x + 5)(x + 2)$

You can multiply using FOIL or by simply distributing each term in the first polynomial to each term in the second polynomial. That is really all that FOIL is – distribution of terms.

First: $x \cdot x = x^2$
Outer: $x \cdot 2 = 2x$
Inner: $5 \cdot x = 5x$
Last: $5 \cdot 2 = 10$

So, $(x + 5)(x + 2) = x^2 + \underbrace{2x + 5x} + 10$

$$= x^2 + \quad 7x \quad + 10$$

3. $(x + 2)(x^2 + 4x - 1) = x(x^2 + 4x - 1) + 2(x^2 + 4x - 1)$

$$= x^3 + 4x^2 - x + 2x^2 + 8x - 2$$

$$= x^3 + 6x^2 + 7x - 2$$

The process of distributing terms still applies here. We don't call it FOIL anymore because we have more than two terms in one of our polynomials, but the idea is the same – just distribute each term in the first polynomial to each term in the second polynomial.

4. $(x^2y + y - 2)(xy^2 + x - 1)$

$= x^2y(xy^2 + x - 1) + y(xy^2 + x - 1) - 2(xy^2 + x - 1)$

$= x^3y^3 + x^3y - x^2y + xy^3 + xy - y - 2xy^2 - 2x + 2$

There are no "like" terms to combine here.

5. $(x + 3)(x - 3) = x^2 \underbrace{-3x + 3x}_{0} - 9$

$= x^2 - 9$

Notice that the middle terms drop out when we are multiplying two binomials which look alike except for the signs between the terms. This is important to note because later when we are factoring, we will use this fact to factor binomials of the form $a^2 - b^2$ by going backwards using the Difference of Squares formula.

Difference of Squares Formula: $(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a^2} \underbrace{-\boldsymbol{ab} + \boldsymbol{ab}}_{\mathbf{0}} - \boldsymbol{b^2}$

$= \boldsymbol{a^2} - \boldsymbol{b^2}$

We need to be careful in the next example not to distribute exponents over addition! This is a common error, but we know it does not work with numbers, so it will not work with variables, either (since variables just represent numbers). For example, $(2 + 3)^2 \neq 2^2 + 3^2$. Work it out and see for yourself.

6. $(x + 4)^2 = (x + 4)(x + 4)$

$= x^2 \underbrace{+4x + 4x} + 16$

$= x^2 \quad + 8x \quad + 16$

When you are squaring a binomial, you can either FOIL it out or use a formula called the Binomial Square Formula. To use it, you can square the "front", square the "back", and then double the product to get the middle term.

Binomial Square Formula: $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$
$\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$

7. $(3y-8)^2 = 9y^2 - 48y + 64$

Using the formula:

Square the "front": $a = 3y \;\rightarrow\; a^2 = (3y)^2 = 9y^2$

Square the "back": $b = 8 \;\rightarrow\; b^2 = 8^2 = 64$

Double the product: $2ab = 2(3y)(-8) = -48y$

So, $(3y-8)^2 = 9y^2 - 48y + 64$

Alternatively, we could have FOILed this out:

$$\begin{aligned}(3y-8)^2 &= (3y-8)(3y-8)\\ &= 9y^2 \underbrace{-24y - 24y} + 64\\ &= 9y^2 \quad -48y \quad + 64\end{aligned}$$

8. $(x-y)(x+y)(x^2+y^2)$

Multiply using difference of squares:

$$\underbrace{(x-y)(x+y)}(x^2+y^2)$$

$$= \quad (x^2-y^2)(x^2+y^2)$$

Multiply again using difference of squares:

$$= \qquad x^4 - y^4$$

9. $x(x-1)(3x+2) - (2x+5)(x-2)$

   This example has more than one operation. There is multiplication and subtraction here. The same order of operations will apply since we know variable just represent numbers and so do variable expressions. So we will do our multiplications first and then perform the subtraction.

$$\underbrace{x(x-1)(3x+2)} - \underbrace{(2x+5)(x-2)}$$

$$= (x^2 - x)(3x+2) - (2x^2 - 4x + 5x - 10)$$

$$= (3x^3 + 2x^2 - 3x^2 - 2x) - (2x^2 + x - 10)$$

$$= 3x^3 - x^2 - 2x - (2x^2 + x - 10)$$

$$= 3x^3 \underbrace{-x^2} \overbrace{-2x} \underbrace{-2x^2} \overbrace{-x} + 10$$

$$= 3x^3 - 3x^2 - 3x + 10$$