Applied Mathematics

Third Edition

By Jim Matovina & Ronald Yates

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PREFACE

This book represents the culmination of efforts to provide a math course to students seeking to earn Associate of Applied Science (AAS) and/or Associate of General Studies (AGS) degrees at the College of Southern Nevada. The content in the course was assembled specifically on the desires and requests of the various Program Directors and Department Chairs for the departments offering those degrees. The course, MATH 104B is intended to teach fundamental mathematical concepts and every-day uses of mathematics AAS and AGS degree-seeking students may encounter in their respective professions and lives.

MATH 104B COURSE DESCRIPTION & OUTCOMES

Course Description

Emphasizing applications, topics include arithmetic, prealgebra, graphing, geometry, finance, probability & statistics. Course is only applicable for AAS and AGS degrees, and is not transferable for credit.

Course Outcomes

Upon completion of this class, students should:

- Be able to participate in a mathematically relevant conversation.
- Be able to accurately perform elementary arithmetic computations.
- Be able to correctly simplify simple algebraic expressions in one variable.
- Recognize and understand various problem-solving techniques.
- Recognize and understand various topics of basic geometry.
- Solve problems involved with the topics of consumer math.
- Recognize and solve elementary probability problems.
- Recognize and compute elementary statistical calculations.
- Have an ability to apply and extend these concepts.

TO THE INSTRUCTOR

This text is intended for a terminal course, specifically designed for Associate of Applied Science (AAS) and Associate of General Studies (AGS) degrees at the College of Southern Nevada. The course does not have any prerequisites and it does not satisfy any prerequisites to other courses. It cannot be applied to a general Associate of Arts (AA) or an Associate of Science (AS), and it will not be transferable to another school in hopes of counting it as a math component for a bachelor's degree. Thus, we are given a fair amount of flexibility in what and how we teach, and that is reflected in the survey nature of the course.

In all the material, you should stress practical applications over symbolic manipulations, but that does not mean you should totally discount the algebra. Instead, help the students learn where, when, why, and how the mathematics will help them in their lives.

TO THE STUDENT

In many courses, memorization is key to performing well. Students who can accurately repeat lists of historical facts, state capitals, spelling words, or sets of basic instructions often get through many courses with high marks. Unfortunately, math is not a subject that falls into that category. Math must be understood. Sure, some fundamental facts – such as the multiplication tables and place value names – should be memorized, but beyond that, the difficulties many people face are due to misguided attempts to memorize everything. Simply put, there is an enormous difference between memorizing facts and understanding concepts.

Knowing how to simplify a linear expression is very different from knowing the capital of South Dakota. You can attempt to memorize every equation and expression you see in this book, but a slight change to a single number can produce something you have never seen before. If you have spent the last 10-15 years (or more!) struggling with your math skills, one course is not very likely to transform your educational experience. If, however, you set aside attempts to memorize everything and focus on understanding a thought process, you can begin to appreciate and even enjoy the problem-solving experience.

Next, one of the goals in assembling this book was to keep the cost very low. As a result, you have a consumable-style workbook you will not be able to sell back to the bookstore at the end of the term. Take advantage of this and write in your book; it is yours to keep.

Finally, we recognize students taking this course are seeking an AAS/AGS degree and need only this one math class in order to graduate. Additionally, in many cases, those same students often wonder why they need to take this class in the first place. All of us – yes, including your instructor – have sat in a math class at one time or another and thought, “When are we ever going to use this stuff?” Well, this is the class where we tell you. Enjoy the ride.

NEW TO THE THIRD EDITION

Changes to Chapter 1

Section 1.1

- The material on binary and hexadecimal numbers was moved to Section 1.2.
- The material on leading digit estimation was eliminated.
- The material on regular estimation techniques was deemphasized.
- The examples and exercises were updated, accordingly.

Section 1.2

- The material on fraction arithmetic was eliminated.
- The material on binary and hexadecimal numbers was expanded, and became the sole focus of the section.
- The examples and exercises were updated, accordingly.

Section 1.3

- The examples and exercises were updated to avoid fractional answers.

Section 1.4

- Material was added to discuss multiplication by a fraction.
- The examples and exercises were updated, accordingly.

Section 1.5

- The “Problems” were retitled to “Examples.”
- The solutions for the examples were moved so they follow the corresponding examples.
- Exercises were updated.

Changes to Chapter 2

- Chapter was retitled to “Geometry & Measurement.”

Section 2.1

- The material on plotting points and graphing lines was eliminated.
- The material on slope was merged into Section 2.5.
- The material about the metric system and dimensional analysis was moved to Section 2.1 from Section 2.2.
- The material on measurement conversions was updated.
- The use of unit fractions was emphasized for conversions.
- Material was added about online converters.
- The examples and exercises were updated, accordingly.

Section 2.2

- The material introducing polygons and circles was moved to Section 2.2 from Section 2.3.
- More facts about polygons were added.
- The material on tilings/tessellations was eliminated.
- Material identifying boxes, pyramids, and cylinders was added.
- The examples and exercises were updated, accordingly.

Section 2.3

- With the material on the introduction to polygons and circles moved to Section 2.2, this section now focuses on perimeter, area and volume.
- The material on finding the perimeter and area of composite figures was expanded.
- Material covering the surface area of boxes, pyramids, and cylinders was added.
- Material covering the volume of boxes, pyramids, and cylinders was added.
- The examples and exercises were updated, accordingly.

Section 2.4

- The examples and exercises were updated.

Section 2.5

- The material about the distance formula was eliminated.
- Distances are computed exclusively with the Pythagorean theorem.
- Material on finding the slope given the rise and the run was moved to Section 2.5 from Section 2.1.
- The material on finding the slope between two points was eliminated, as was the slope formula.
- The examples and exercises were updated, accordingly.

Changes to Chapter 3

Section 3.1
- The material on percent change was eliminated.
- Material on monthly budgeting was added.
- The examples and exercises were updated, accordingly.

Section 3.2
- Examples were added, and the exercises were updated, accordingly.

Section 3.3
- Credit card balance calculations are now done with line-by-line computations, instead of using tables.
- Examples were added, and the exercises were updated, accordingly.

Section 3.4
- Examples were added, and the exercises were updated, accordingly.

Section 3.5
- The monthly mortgage payment formula and spreadsheet references were eliminated.
- The use of an online mortgage calculator was changed from a dedicated website to using the mortgage calculator embedded in Google search results.
- Mortgage calculations are now rounded to the whole dollar instead of the nearest cent.
- Examples were added, and the exercises were updated, accordingly.

Changes to Chapter 4

Section 4.1
- Examples were added, and the exercises were updated, accordingly.

Section 4.2
- The material on sports wagering was eliminated.
- Examples were added, and the exercises were updated, accordingly.

Section 4.3
- Examples were added, and the exercises were updated, accordingly.

Section 4.4
- The material on frequency tables was eliminated.
- The material on GPA calculations was refined.
- Examples were added, and the exercises were updated, accordingly.

Section 4.5
- The material on stem-and-leaf plots, histograms and frequency tables was eliminated.
- The material on bar graphs, line graphs, and pie charts was expanded and refocused, to emphasize the construction of, and appropriate uses for, each.
- Examples were added, and the exercises were updated, accordingly.

Changes Throughout the Text

- The spacing between sentences was reduced from two to a single space.
- A surrounding border was added around each example.
- Key Concepts were placed inside blue boxes.
- Common Mistakes were placed inside orange boxes.
- A "Using Your Calculator" passage was added to many sections.
- Internet/cell phone applications were added, where appropriate.
- The pictures and images were changed to those found in the Public Domain.
- ADA issues, including the alt text for images and links, were addressed.
- Most of the exercise answers were expanded into solutions.
- Image credits were added to the end of each chapter.
- Mistakes/typos were corrected. (Hopefully, none were added.)

ACKNOWLEDGEMENTS

Gratitude is given to the following individuals for their tireless and diligent efforts in reviewing and contributing to this book and the corresponding course.

> Denny Burzynski, Dennis Donohue, Billy Duke, Michael Greenwich, JoAnn Friedrich, Bill Frost, Aaron Harris, Eric Hutchinson, Amin KM, James Lee, Andrzej Lenard, Garry Knight, Mason Matovina, Sherry Norris, Ingrid Stewart, Stan VerNooy, Patrick Villa, and Tityik Wong.

A special "Thank You" is given in advance to all the students and teachers who will help improve this book. Being written and published by a math professor, any volume of work of this size is bound to have typos. If you happen to stumble across what you believe to be an obvious mistake, please take a few moments and send it to jim.matovina@csn.edu.

Here's hoping you have as much fun reading it as we did putting it together.

Jim Matovina & Ronnie Yates

CHAPTER 1: ARITHMETIC & PREALGEBRA

A few years ago, a class was presented with the following problem: “A farmer looks across his field and sees pigs and chickens. If he counts 42 heads and 106 feet, how many pigs are in the field?” One student, instead of attempting to answer the question, sent an e-mail to the instructor stating, “I feel this question is unfair and misleading because pigs do not have feet. They have hooves.”

In this class, and in all math classes for that much, we are not trying to trick you. Although you will be presented with some challenging questions, be assured they are there to make you apply and extend your thought processes. Don't waste time and effort looking for technicalities that might invalidate a question. Instead, take the questions at face value, and spend your time trying to solve them.

Before we can get into some of the more practical and applied math, we will spend a big chunk of this first chapter reviewing some elementary concepts and properties.

CHAPTER OUTLINE AND OBJECTIVES

Section 1.1: Rounding, Decimals & Order of Operations

A. Understand the difference between estimation and rounding.
B. Be able to round numbers to identified place values.
C. Be able to perform decimal arithmetic.
D. Be able to simplify arithmetic expressions according to the correct order of operations.

Section 1.2: Binary & Hexadecimal Numbers

A. Recognize binary and hexadecimal numbers.
B. Be able to convert binary and hexadecimal numbers to their decimal equivalents.
C. Be able to convert decimal numbers to their binary or hexadecimal equivalents.

Section 1.3: Expressions & Equations

A. Be able to simplify algebraic expressions.
B. Be able to apply the Addition Property of Equality.
C. Be able to apply the Multiplication Property of Equality.
D. Be able to solve simple linear equations.

Section 1.4: Fair Division

A. Perform fair division using the Divider-Chooser Method.
B. Perform fair division using the Method of Sealed Bids.

Section 1.5: Problem Solving

A. Understand the general problem-solving process.
B. Be able to apply various problem-solving techniques.

1.1: ROUNDING, DECIMALS & ORDER OF OPERATIONS

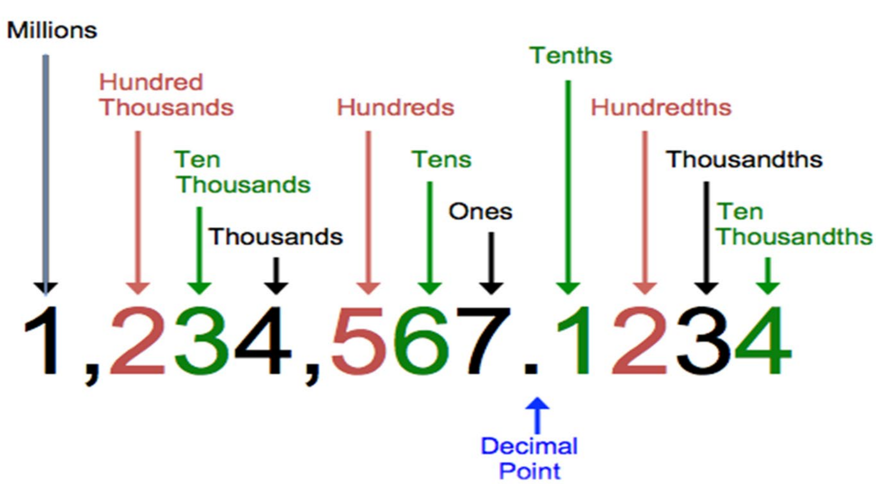
All too often, we reach for a calculator when faced with a multi-step computation. Granted, there is nothing wrong with wanting to use a calculator to ensure a little bit of accuracy, but, unfortunately, many people blindly trust the calculator's answer without considering the possibility of a mistaken entry. It is really easy to miss a key or accidentally type the wrong number. Some very basic estimation done before we start hitting buttons can save us a lot of embarrassment in the long run.

Addition, subtraction, multiplication and division performed with decimals is the same as with whole numbers, with a greater emphasis on the placement of the decimal point. For addition and subtraction, set up the problem vertically – being careful to align the decimal points in the given numbers. For multiplication, the total number of decimal places appearing in the numbers being multiplied together has to be the same number of decimal places in the product. When performing long division with a decimal divisor, we move the decimal points in the divisor and the dividend, so that the division is being done by a whole number. We also need to maintain decimal point alignment in the quotient, as well. Before we get too far into a review of decimal arithmetic, we need to first discuss place values, estimation, and rounding.

Place Values

Place values extend forever in both directions, but we will concentrate on the places close to the decimal point. Immediately to the left of the decimal point is the **ones** place (if we are asked to round to the nearest **whole number**, we round to the ones place). To its left is the **tens** place, then, continuing to the left, we have the **hundreds**, **thousands**, **ten thousands**, **hundred thousands**, and **millions** places.

To the right of the decimal point, it is important to note there is no "oneths" place. The first place value to the right of the decimal point is the **tenths** place. Moving to the right, we have the **hundredths**, **thousandths**, and **ten thousandths**. In fact, skipping the ones place, the place values to the right of the decimal point are a mirror image of those on the left. On the right, however, we add the **–ths** suffix onto each name.



Estimation vs. Rounding

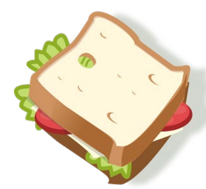
What is the difference between precision and accuracy? Precision refers to the detail of the answer. Accuracy is concerned with its correctness. If I say the current time is 47 minutes and 53.56 seconds past one in the afternoon, you could say my statement is very precise. However, if it is actually 5:30 in the morning, my statement is not very accurate.

Many people have some misconceptions when it comes to when and how to correctly round numbers. Furthermore, some people believe that rounding and estimating are the same thing. They are not. Just like measurements can be very precise, but not very accurate, there are fundamental differences between rounding and estimating.

Estimation

Estimation is done to determine an approximate value of a calculation. Estimations do not have to be precise; in fact, they are much easier to handle when they are less precise. There are many different ways to perform an estimation, and it is not uncommon for different people to estimate answers to different amounts. We do, however, hope a stated estimation is reasonably accurate. Estimations are done for our benefit and our benefit alone, just to get a rough idea of what the answer should be. Furthermore, they should be done quickly and, preferably, in our heads.

Example 1: You are grocery shopping and realize you only have $10. You want to buy a gallon of milk for $3.69, a bag of chips for 99¢, a sandwich from the deli for $3.99. Quickly estimate the total cost of your items to see if you will have enough money.



You can estimate the milk to cost $4, the chips to cost $1, and the sandwich to cost $4, making the total $9. $10 should be enough.

Rounding

Estimations can be done to any place value we desire. **Rounding**, however, involves a specified place value. We MUST have a place value to which to round. Without a specified place value, it is incorrect to assume one out of convenience.

If there is no directive to round to a specified place value, it is incorrect to do so.
After all, if no place value is mentioned, we cannot assume one out of convenience.

The formal procedure for the traditional rounding of a number is as follows.

1. First, determine the round-off digit, which is the digit in the specified place value column.

2. If the first digit to the right of the round-off digit is less than 5, do not change the round-off digit, but delete all the remaining digits to its right. If you are rounding to a whole number, such as tens or hundreds, all the digits between the round-off digit and the decimal point should become zeros, and no digits will appear after the decimal point.

3. If the first digit to the right of the round-off digit is 5 or more, increase the round-off digit by 1, and delete all the remaining digits to its right. Again, if you are rounding to a non-decimal number, such as tens or hundreds, all the digits between the round-off digit and the decimal point should become zeros, and no digits will appear after the decimal point.

4. For decimals, double-check to make sure the right-most digit of the decimal falls in the place value column to which you were directed to round, and there are no other digits to its right.

Be sure to pay attention to the directive, and for rounding to place values to the right of the decimal point, make sure the last digit of the answer lies in the designated place value. That is, if we are rounding to the tenth, the last digit of the answer must lie in the tenths place.

When rounding to place values to the left of the decimal point, all the digits between the designated place value and the decimal point need to be zeros. For example, 7956 rounded to the nearest thousand yields an 8 in the thousands place and 0s in the hundreds, tens, and ones places: 8000.

Example 2: Round 103.4736999 to the nearest tenth.

Begin by recognizing the 4 is in the tenths place. The 7 immediately to its right indicates we are to change the 4 to a 5, and remove the rest of the digits. Thus, to the tenth, the value is 103.5.

Example 3: Round 103.4736999 to the nearest hundredth.

The 7 is in the hundredths place. The 3 immediately to its right indicates we are not to change the 7, and remove the rest of the digits. Thus, to the hundredth, the value is 103.47.

Example 4: Round 103.4736999 to the nearest hundred.

The 1 is in the hundreds place. The 0 immediately to its right indicates we are not to change the 1, and all the digits between the 1 and the decimal point are to become 0's. Thus, to the hundred, the value is 100.

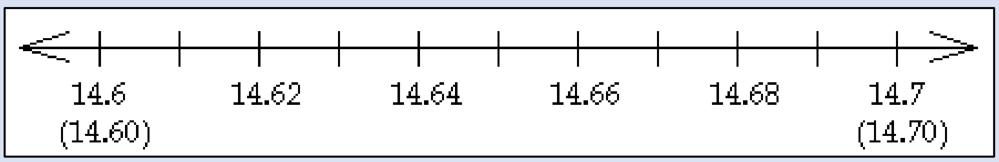
Don't Get Trapped Memorizing Instructions

We cannot let ourselves get trapped in a process of trying to memorize a set of instructions. Yes, if followed correctly, instructions can be quite valuable, but when we blindly try to stick to a formal, rigid process, we often lose sight of logic and reason. For many students, rounding numbers is a classic example of this.

If, for a second, we set aside the traditional process and understand that "rounding a number to the tenth" can be thought of as "identifying the number ending in the tenths place that is closest to the given number," we can gain a better understanding of what it means to round.

For example, if we are tasked with rounding 14.68 to the nearest tenth, we could begin by identifying the consecutive numbers ending in the tenths place (remember, for decimals, the right-most digit of the answer must fall in the stated place value) that are immediately less than and immediately greater than 14.68. Those two numbers are 14.6 and 14.7. Then, we can simply ask ourselves, "Is 14.68 closer to 14.6 or 14.7?"

If we look at a number line…



Treating 14.6 and 14.7 as 14.60 and 14.70, respectively, makes it easy to see 14.68 is closer to 14.7, than it is to 14.6. The big thing to remember is the last digit of our rounded answer MUST fall in the place value to which we are rounding. In other words, the answer needs to be 14.7, and not 14.70.

Example 5: Round 13.99 to the nearest whole number.

For the whole number, or ones place, begin by recognizing the 3 is in the ones place. The whole numbers immediately before and after the given number are 13 and 14. 13.99 is closer to 14.

Example 6: Round 13.99 to the nearest hundred.

For the hundred, begin by recognizing a 0 is in the hundreds place – even though it is not written. Counting by 100s, the values immediately before and after 13.99 are 0 and 100. Since 13.99 is closer to 0, to the hundred, 13.99 rounds to 0.

In the previous example, even though the 0 is, technically, in the ones place, we do not write the answer as "000;" we just write 0. 0 is the only exception to the rightmost digit rule.

When to Round and When NOT to Round

Throughout this book, and all of mathematics, for that much, we will see many directives to "round to the nearest hundredth (or other place value), when necessary." The important part of that directive is the "when necessary" part. How are we supposed to know when rounding is necessary and when it isn't?

First, if we are given the absolute directive "round to the nearest hundredth" (or another place value) we MUST round to that specified place value. Keep in mind, when we are working with money, the underlying – but often unstated – directive is to round to the nearest cent.

Then we run into the "when necessary" scenario. This only applies to **non-terminating decimals**. So, just what is a non-terminating decimal?

When we change a fraction to a decimal, we divide the numerator by the denominator. To write 2/5 as a decimal, we divide 2 by 5 to get 0.4. When doing such a division, if we come to a place where the division ends, we have a **terminating decimal**. If the division does not end, it will begin to repeat. 5/9 as a decimal is 0.55555… The set of three decimal points at the end is called an **ellipsis**, and signifies we are to continue the pattern that has been established. A decimal that repeats a pattern is called a **repeating decimal**.

There are also decimals that do not terminate and do not repeat. The well-remembered number from geometry class, π (pi) is one of them. Even though we typically use the value of 3.14, the value of π is closer to 3.141592654. In fact, the decimal part goes on forever and does not repeat. The technical name for this group is **transcendental numbers**, but for our purposes, we will group the transcendental numbers and repeating decimals together and refer to them as **non-terminating decimals**.

Now, let's go back to our "when necessary" dilemma. If the decimal terminates, there usually is no need to round it. In fact, if we round it, our answer is less accurate than it could be. In other words, 1/32 = 0.03125, exactly. If we round that decimal to the hundredth, to 0.03, the value is no longer equal to 1/32. Sure, it's close, but it is NOT equal to 1/32; it's actually 3/100.

Non-terminating decimals are the ones to which we need to pay attention. If we dismiss the use of the ellipsis, it is impossible to write 1/3 as a decimal. 0.33 = 33/100, but 1/3 = 33/99. 0.33333 = 33333/100000, but 1/3 = 33333/99999. 0.333333333333333 is closer, but as soon as we stop writing 3's, we no longer have exactly 1/3. Thus, in order to save us from writing 3's indefinitely, it is necessary to round it to a specified place value.

Keep in mind; a calculator does not display an ellipsis. If you change 5/9 to a decimal, the calculator may display the "final" digit as a 6: 0.555555556. Calculators typically round to the number of digits they can display on their screens. Some calculators may just truncate the decimal to the screen size, and others may actually hold an extra three to five digits of the decimal in memory without displaying them – this is actually the smarter version, as a greater amount of precision leads to a greater degree of accuracy.

If the decimal terminates within three or four decimal places, there may be no need to round. If it extends past four decimal places AND we have a specified place value to which to round, then go ahead and do so. Remember, though, there MUST be a designated place value.

One of the few exceptions to this rule involves money. As mentioned earlier, if money is involved, unless we are specifically told to round to a less precise value, such as the nearest dollar, we should always round to the nearest cent.

Adding and Subtracting Decimals

Adding and subtracting decimals is almost exactly like doing so with whole numbers. The only extra step is the proper placement of the decimal point. With addition and subtraction, we like to set up the problem vertically and align the decimal points in the numbers. We also may need to include extra zeros in subtraction problems to ensure the decimals have the same place value agreement.

Example 7: Subtract: 19.8 – 7.67

We may be more comfortable performing the subtraction by aligning the numbers vertically and writing 19.8 as 19.80. If so, be sure to line up the decimal points.

$$\begin{array}{r} 19.80 \\ -\ 7.67 \\ \hline \end{array}$$

In the bottom number, since the 7 in the hundredths place is less than the 0 appearing directly above it, we have to borrow. Once everything is set up, perform the subtraction in each column, and bring the decimal point straight down into the answer.

$$\begin{array}{r} {\scriptstyle 7\ 10} \\ 19.\cancel{8}\cancel{0} \\ -\ 7.67 \\ \hline 12.13 \end{array}$$

Multiplying Decimals

In multiplication, we multiply the numbers and then add together the number of decimal places appearing in the two numbers being multiplied together. That total tells us how many decimal places are in the answer.

Dividing Decimals

With division involving decimals, we all learned a lengthy process involving the scaffolding method of long division. Although certainly worth learning, processes like that often see students resort to memorization without taking the time to understand the underlying concepts. Remember, division with decimals is just like division with whole numbers. The only additional step is making sure the decimal is properly placed in the answer. Using a calculator can make the process faster, but we need to be sure to not place blind faith in what the calculator tells us – it is far too easy to make a mistake hitting small buttons with large fingers. ☺

In a division problem, the number being divided is called the **dividend**, the number doing the dividing is the **divisor**, and the answer is called the **quotient**. Dividing by the number 1 does not change the dividend. Dividing by a number greater than 1 makes the quotient smaller than the dividend. Inversely, dividing by a number less than 1 makes the quotient greater than the dividend.

We all know 14 divided by 2 is 7. So, what is 14 divided by 0.2? If we ignore the decimal point for a minute, we are still dividing the number 14 by the number 2. The presence of the decimal point will not change that. The presence of the decimal point in the divisor will ONLY change the location of the decimal point in the quotient. Since the divisor, 0.2, is less than 1, the quotient must be larger than the dividend. The decimal point was moved one place in changing the 2 to the 0.2, so we make the quotient of 7 larger by moving the decimal one place to make it 70.

Order of Operations

When several people are asked to simplify the same multi-step arithmetic expression we need to be sure they all perform the individual calculations in the same order. Otherwise, different people will get different values. For example, let's say Bill and Ted are asked to simplify the expression $3 + 4 \times 5$. Bill adds the $3 + 4$, and then multiplies that result by 5 to get a total of 35. Ted performs the product first, and then adds in the 3 to get a total of 23. Who is right?

There are a lot of different calculators on the market, and, believe it or not, some of them do not correctly simplify multi-step arithmetic expressions. Essentially, this is the basic difference between standard and scientific calculators. The typical standard calculator is not programmed to follow the correct order of operations. In other words, if we typed in $3 + 4 \times 5$ into a standard calculator, it will actually tell us the result is the incorrect value of 35. If we typed the same expression to a scientific calculator, it will perform the multiplication first, and will give us the correct result of 23.

Arithmetic, as we know it, all started with addition (and subtraction). Repeated addition led to multiplication (and division), and repeated multiplication led to the use of exponents. To circumvent that hierarchy, we use parentheses or other grouping symbols. Writing all that into a list form, we see the traditionally presented "order of operations."

Arithmetic Order of Operations:

1. All operations contained within parentheses or other grouping symbols, such as brackets [ ], or braces { }, should be done first.

2. Secondly, simplify all expressions containing exponents.

3. Multiplication and division are done next, as we come to them going from left to right.

4. Addition and subtraction are done last, again, as we come to them going from left to right.

To help remember this order, many students like to memorize a cute little acronym like **PEMDAS** (Parentheses, Exponents, Multiplication, Division, Addition, Subtraction). Be careful! If you do not realize multiplication and division are done as we come to them going from the left to the right, you may fall into the trap of thinking multiplication always precedes division – it does not. The same hold true for addition and subtraction.

Example 8: Simplify: $5 + 6.2 \times 1.7$

Start by multiplying 6.2×1.7. Then add 5 to that result.

$5 + 6.2 \times 1.7$

$5 + 10.54$

15.54

Example 9: Simplify: $13 - 5 + 6$

Remember, perform the addition and subtraction as we come to them going from left to right. That means, $13 - 5$ must be done first.

$13 - 5 + 6$

$8 + 6$

14

Example 10: Simplify: $13 - (5 + 6)$

Since $5 + 6$ is in parentheses, it should be done first.

$13 - (5 + 6)$

$13 - 11$

2

Example 11: Simplify $5 \times (2+3)^4 - (6-4) + 1$

Perform the operations within the parentheses first. Then apply the exponent. After that, perform the multiplication. Finish by performing the addition and subtraction going from left to right.

$5 \times (2+3)^4 - (6-4) + 1$

$5 \times (5)^4 - (2) + 1$

$5 \times 625 - 2 + 1$

$3125 - 2 + 1$

$3123 + 1$

3124

Be sure to pay attention to the location, inclusion, and even the exclusion of the grouping symbols. Look at the slight difference between the following two examples.

Example 12: Simplify: $42 \div 3(2)$

In this case, the parentheses only encase the 2. Thus, the division and multiplication are done first - going from left to right.

$42 \div 3(2)$

$14(2)$

28

Example 13: Simplify $42 \div [3(2)]$

In this case, the brackets encase the operation "3(2)" meaning it must be done first.

$42 \div [3(2)]$

$42 \div [6]$

7

Round ONLY as the Last Step

Our discussion of decimals is not complete without a mention of the consequences of rounding too early. If we are going to round, doing so is ALWAYS the very last step in a problem. One very common mistake in multi-step problems is to round at an intermediate step. This mistake is often compounded by the fact that the student is using a calculator.

In any multi-step simplification in which we are directed to round, the rounding is always the very last step.

Using Your Calculator

Calculators are extremely useful tools. You may have noticed a lack of examples in the presentation of the material covering addition, subtraction, multiplication and division of decimals.

Do realize, however, the calculator is only as accurate as the one using it. If you are prone to what a lot of students refer to as "fat fingers," you are advised to repeat each calculator computation at least twice.

Example 14: Use a calculator to compute: $1000(1 + 0.14/12)^{24}$. Round your answer to the nearest hundredth.

The order of operations says we do the 0.14/12 first. Many students will use a calculator and see that 0.14/12 = 0.011666… and "round" it to 0.012. Then, if we add 1, raise the 1.012 to the 24th power and multiply the result by 1000, we get a total of 1331.4728…, or 1331.47 when we round the answer to the hundredth.

In actuality, $1000(1 + 0.14/12)^{24} = 1320.9871\ldots$, or 1320.99 to the nearest hundredth. By "rounding" in an intermediate step, our answer of 1331.47 was incorrect. If this were money, our previous calculation was off by over $10, but, in a similar problem could very easily have turned out to be much greater.

In the previous example, the key to avoiding early rounding mistakes was to use the calculator effectively. To do so, leave the entire decimal representations in the calculator as much as possible. Also, if we make use of the parentheses keys, we should never have to hit the = button more than once in any multi-step computation.

Let's do the same example again, but this time, we will use the calculator's features a little better. On your calculator locate the exponentiation key and the parentheses keys. If you need help, ask your instructor.

Example 15: Compute: $1000(1 + 0.14/12)^{24}$ hitting the equals key on your calculator only once. Round your answer to the nearest hundredth.

Depending on your calculator, the exponentiation key, x^y may look like ^.

Literally type: 1000 × (1 + 0.14 ÷ 12) x^y 24 =.

You should get 1320.9871…, which you will manually round to 1320.99.

If you don't have a scientific calculator, you can usually purchase one for $10-20 in a number of locations. iPhone users can bring up the system calculator and rotate the phone to landscape mode. You can also download a free calculator app for your cell phone that will work just as easily. To help with material covered later in this book, look for a scientific calculator or an app that allows for permutations and combinations (the keys would be labeled "nPr" and "nCr").

You can pay over $100 for a graphics calculator, but that tool is not necessary for this material.

Section 1.1 Exercises

For Exercises #1-5, read each problem carefully and estimate the answer.

1. The distance between Joe's house and school is 9.25 miles, and Joe goes to school 4 days a week. Estimate the total number of miles Joe drives between home and school in one month.

2. Using the information from the previous exercise, let's assume Joe gets 29 miles per gallon. If gas is $2.89 per gallon and Joe only uses his car to go to and from school, approximately how much money does he spend per month on gas?

3. Julia purchased a new car and traveled 356 miles before refueling. If she needed 15.6 gallons of gas to fill the car's tank, estimate her gas mileage.

4. Using the information in the previous exercise, if the cost of the gas is $3.48 per gallon, estimate the total cost of the gas.

5. If the life span of a light bulb is 2500 hours, approximately how many weeks can you keep this light on 24 hours a day?

6. A sign mounted on a panel in an elevator reads, "Capacity: 3300 pounds (17 people)." Do you think the elevator can support 22 first grade children? Why?

7. For the number 9185, round to the following place values.
   a. Hundreds
   b. Tens
   c. Ten-thousands

8. For the number 2555, round to the following place values.
   a. Hundreds
   b. Tens
   c. Ten-thousands

9. For the number 205.687, round to the following place values.
   a. Hundreds
   b. Hundredths
   c. Ten-thousandths

10. For the number 987.65432, round to the following place values.
    a. Hundreds
    b. Hundredths
    c. Ten-thousandths

11. For the number 9185.444445, round to the following place values.
    a. Hundreds
    b. Hundredths
    c. Ten-thousandths

12. For the number 3.9999999, round to the following place values.
    a. Hundreds
    b. Tens
    c. Whole Number

13. For the number 8255.687, round to the following place values.
    a. Hundreds
    b. Hundredths
    c. Thousandths

14. Add or subtract, as indicated.
    a. $5.12 + 3.9$
    b. $75.12 - 18.973$

15. Add or subtract, as indicated.
    a. $31 - 8.5412$
    b. $32.95 + 27.444$

16. Multiply, as indicated.
    a. 7.12×1.9
    b. 35.2×28.471

17. Multiply, as indicated.
    a. 11×2.5492
    b. 0.32×9.1

18. Divide, as indicated.
    a. $5.12 \div 2$
    b. $8.5412 \div 2.5$

19. Simplify the following expressions.
    a. $2.5 + 3.2 \times 2.7$
    b. $43 - 15 + 16$

20. Simplify the following expressions.
    a. $6 \times (2 + 1)^3 - (9 - 4) - 8$
    b. $6^2 \div 4 - (3 + 5)$

21. Simplify the following expressions.
    a. $(4 + 6)^2 \div 2 \times 5 + 7$
    b. $(9^2 - 63) \times (3 + 5) \div 36$

Section 1.1 Exercise Solutions

NOTE: For Exercises #1-5, you were asked to provide an estimation. Since estimations may differ from person to person, the answers may vary.

1. 10 miles/trip × 2 trips/day × 4 days/week × 4 weeks/month = 320 miles/month.

2. Approximately 300 miles/month ÷ 30 miles/gallon is about 10 gallons of gas per month. Gas is about $3.00 per gallon, so 10 gallons will cost about $30.

3. 300 mi/15 gal = 20 mpg.

4. Gas is about $3.50 per gallon, so the total cost is about $52.50.

5. 2400 hours/24 hr per day means the bulb will last a little more than 100 days. Rather than dividing 100 by 7, if we notice 98/7 = 14, we can say the bulb will last a little more than 14 weeks. Since we underestimated twice, (we took 2500 to 2400 and 100 to 98), it may even be more accurate to say the bulb will last about 15 weeks.

6. Since a first grader weighs less than 100 pounds, 22 of them will be less than 2200 pounds. So, yes, the elevator will hold 22 first grade children.

7. a. 9200 b. 9190 c. 10,000

8. a. 2600 b. 2560 c. 0

9. a. 200 b. 205.69 c. 205.6870

10. a. 1000 b. 987.65 c. 987.6543

11. a. 9200 b. 9185.44 c. 9185.4444

12. a. 0 b. 0 c. 4

13. a. 8300 b. 8255.69 c. 8255.687

14. a. 9.02 b. 56.147 c. 22.4588

15. a. 22.4588 b. 60.394

16. a. 13.528 b. 1002.1792

17. a. 28.0412 b. 2.912

18. a. 2.56 b. 3.41648

19. a. $2.5 + 3.2 \times 2.7$
   $2.5 + 8.64$
   11.14

   b. $43 - 15 + 16$
   $28 + 16$
   44

20. a. $6 \times (2 + 1)^3 - (9 - 4) - 8$
$6 \times 3^3 - 5 - 8$
$6 \times 27 - 5 - 8$
$162 - 5 - 8$
$157 - 8$
149

b. $6^2 \div 4 - (3 + 5)$
$36 \div 4 - 8$
$9 - 8$
1

21. a. $(4 + 6)^2 \div 2 \times 5 + 7$
$10^2 \div 2 \times 5 + 7$
$100 \div 2 \times 5 + 7$
$50 \times 5 + 7$
$250 + 7$
257

b. $(9^2 - 63) \times (3 + 5) \div 36$
$(81 - 63) \times (3 + 5) \div 36$
$18 \times 8 \div 36$
$144 \div 36$
4

1.2: BINARY & HEXADECIMAL NUMBERS

The number system we are all used to using is technically called the **Hindu-Arabic System**. More commonly, since every number is composed of a combination of ten unique digits (0, 1, 2, 3, 4, 5, 6, 7, 8 & 9), the system is referred to as a base 10 or a **decimal number system**.

Keep in mind, the numbers are built with those digits *and* by their position. Writing a 7 in the hundreds place actually stands for 700, while a 7 in the tens place is just 70. Furthermore, since we use those TEN unique digits, each place value is categorized by a power of 10. In order to gain a better understanding of our base 10, place-value system, let's spend some time discussing what happens if we change the base. Two common non-base 10 systems are binary and hexadecimal systems.

Binary Numbers

A **binary number system** is a base 2 system, and, thus, every number is composed of some combination of two unique digits: 0 and 1. Read that again. Every binary number is a combination of 1s and/or 0s. For example, the decimal number 5 would be written as 101 in binary. We will get to how that is done in a little bit.

It is very important to note there is no "2" character in a base 2 system. Just like there is no "10" character in a base 10 system. "10" is not a single character, anyway. A base 10 system has the characters 1 through 9, and a 0. Similarly, in a base 2 system every number is made up of 1s and 0s.

Believe it or not, binary numbers may actually be used *more* than decimal numbers. Binary numbers are used in the fundamental operations of computers. Every number, character, and symbol we see flashing across our computer screen has a binary number attached to it. For example, the word "Hi" would be encoded as "0100100001101001" in binary coded ASCII (American Standard Code for Information Interchange).

Why binary? Computers function on minute pulses of electricity. If a pulse is absent, we can assign that instance to the 0. When the pulse is present, we can assign that instance to the number 1. Then, a character, such as the $, can be represented by 00100100. A **hertz** (Hz) is the measure of the number of cycles (pulses) a computer can recognize in a second. Mega (M-) is the prefix for a million, so 4 MHz stands for 4,000,000 hertz. Giga (G-) is the prefix for a billion, so a computer with a processor speed of 4 GHz can recognize 4 *billion* cycles in a single second. Even though there are many factors to consider when we are shopping for a fast computer, the processor speed is one of the most important ones.

Converting Binary Numbers to Decimal Numbers

Let's return to how we read and write binary numbers. Think of the place values for the digits in the decimal system for the number 7204, which can be written in expanded form as 7000 + 200 + 0 + 4.

Like the decimal system, the binary system is a place-value system. However, instead of powers of 10 ($10^0 = 1$, $10^1=10$, $10^2 = 100$, $10^3 = 1000$, etc.), each place value corresponds to a power of 2 ($2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4=16$, etc.).

So, how can we tell the difference between the number 100 in a decimal system and the number 100 in a binary system? After all, the binary number 100 is equivalent to 4 in a base 10 system, not one hundred. There are two ways. The first is context; If we are told 100 is a binary number, then there is no argument. The second way to tell is by the use of a subscript following the number. Remember, subscripts are a little smaller and appear slightly below the baseline for the stated number. 100_2 indicates the number is 100 and the base is 2. Likewise, 200_{16} would mean the number is 200 and the base is 16. And, here's the kicker, since base 10 is the common system, the absence of the subscript will mean the number is stated in base 10. That is, we will never see 100_{10}, as it would be written as just 100.

Just like a 7 in the hundreds place in a base 10 system represents 7 hundreds (better known as 700), a 1 in the fours place in a binary system would stand for 1 four, which is just 4.

Example 1: Write the binary number 1010 as a decimal number. (NOTE: Context tells us 1010 is binary.)

$$\begin{aligned} 1010 &= 1\times2^3 + 0\times2^2 + 1\times2^1 + 0\times2^0 \\ &= 1\times8 + 0\times4 + 1\times2 + 0\times1 \\ &= 8 + 0 + 2 + 0 \\ &= 10 \end{aligned}$$

Example 2: Write the binary number 110110 as a decimal number.

$$\begin{aligned} 110110 &= 1\times2^5 + 1\times2^4 + 0\times2^3 + 1\times2^2 + 1\times2^1 + 0\times2^0 \\ &= 1\times32 + 1\times16 + 0\times8 + 1\times4 + 1\times2 + 0\times1 \\ &= 32 + 16 + 0 + 4 + 2 + 0 \\ &= 54 \end{aligned}$$

Some students who are working with binary numbers for the first time have an easier time by noting that the place values "start" on the far right with the ones place. Then, moving leftward, double each place value.

Think about that for a minute. In a decimal system, the far-right place value (before the decimal point) is the ones place, and, moving to the left, we multiply each place value by 10 (ones, tens, hundreds, thousandths, etc.).

So, in the binary number 101, the right-most 1 is in the ones place, the 0 is in the twos place, and the left-most 1 is in the fours place. In other words, we have 1 one, 0 twos, and 1 four. 1 + 0 + 4 = 5.

Example 3: Write the binary number 11001 as a decimal number.

The right-most 1 is in the ones place. Moving left and doubling each place value, we have 0 twos, 0 fours, 1 eight, and 1 sixteen. 1 + 0 + 0 + 8 + 16 = 25.

Example 4: Write the binary number 101100 as a decimal number.

Starting on the right and moving left, we have 0 ones, 0 twos, 1 four, 1 eight, 0 sixteens, and 1 thirty-two. 0 + 0 + 4 + 8 + 0 + 32 = 44.

Converting Decimal Numbers to Binary Numbers

Converting a binary number to a decimal number is pretty straightforward, but going the other way is bit more involved. To do so, we have to look for the powers of 2 that appear in a sum leading up to the given number. If we determine a specific power of 2 is contained in the given number, we indicate there is one of them in the appropriate place value, deduct that power of 2 from the given number, and continue with this process until we account for the sum that gives us the original number.

Remember, the powers of 2 are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4=16$, $2^5 = 32$, $2^6 = 64$, and so on.

As we saw in an earlier example, 54 = 32 + 16 + 0 + 4 + 2 + 0, so let's use that for our next example.

Example 5: Write 54 as a binary number.

$2^6 = 64$, which is greater than 54, so we cannot have a 1 in the sixty-fours place (or any larger place, either).

$2^5 = 32$. So, we begin by noting that we need to have a 1 in the thirty-twos column. Then, subtracting the 32 from 54 we see we have 22 left.

Then we inspect the 22 for powers of 2. $2^4 = 16$, which is less than 32, so we have a 1 in the sixteens place. Also, since 22-16 = 6, we have 6 left to account for.

Continue this process until we have the desired sum, using 0's as place holders for the place values we skip over, all the way down to 1, which is 2^0.

54 = 32 + 16 + 0 + 4 + 2 + 0

= 110110_2 (Remember, the subscript, $_2$ indicates the 110110 is binary.)

Example 6: Write 44 as a binary number.

Remember the multiples of 2: 1, 2, 4, 8, 16, 32, 64, …

64 is too big, so start with 32.

We have 1 in the thirty-twos place, and 44-32 = 12.

12 is smaller than 16, so we have a 0 in the sixteens place.

Continue the process to find 12 is composed of 1 eight, 1 four, 0 two, and 0 ones.

$44 = 32 + 0 + 8 + 4 + 0 + 0$

$\quad = 101100_2$

Example 7: Write 113 as a binary number.

Again, the place values are multiples of 2: 1, 2, 4, 8, 16, 32, 64, 128, 256…

128 is too big, so we start with the sixty-fours place.

$113 \quad = 64 + 32 + 16 + 0 + 0 + 0 + 1$

$\quad\quad = 1110001_2$

Binary numbers can also be useful in date codes. Here is picture of a label on a bottle of beer. To indicate the expiration date, a machine was programmed to place notches along the bottom and left edges of the label. The month and year across the bottom are easy to read.

For the day of the month, the left side of the label is printed with | 1 | 2 | 4 | 8 | 16 |, and the notches appear in the 1, 4 and 16 regions. Although technically printed in reverse order, the notches indicating the day of the month correspond to the binary number 10101, which is equivalent to 21. Putting it all together, in this case, the expiration date for the beer was July 21, 1996.

It is worth noting the next binary place value is 32, and no month has more than 31 days.

Hexadecimal Numbers

A decimal system has ten digits. The prefix dec- means 10. A decade is ten years. A binary system has two digits. The prefix bi- means 2. A bicycle has two wheels.

The prefix hexa- means 6. A hexagon has six sides. When combined with deci-, the prefix hexadeci- means 16. So, a **hexadecimal** (usually called "hex") **numbering system** has exactly sixteen unique single-character digits. The first ten digits are the same as a decimal system: 0, 1, 2, 3, 4, 5, 6, 7, 8 & 9. But, we cannot use "10" as the next digit because it is not a single character. Thus, for the remaining six single-character digits we use the capital letters: A, B, C, D, E & F.

Converting Hexadecimal Numbers to Decimal Numbers

In a hex system, the digit A is equivalent to 10 in a decimal system. Likewise, B is equivalent to 11, C is equivalent to 12, D is 13, E is 14, and F is 15. And, more importantly, since we have 16 unique characters, the place values MUST be powers of 16 ($16^0 = 1$, $16^1 = 16$, $16^2 = 256$, $16^3 = 4096$, etc.).

Example 8: Write the hexadecimal number B3 as a decimal number. (NOTE: Context tells us B3 is a hexadecimal number.)

The B is in the sixteens place and the 3 is in the ones place. Thus, we have "B" sixteens, and 3 ones. Remember, B is worth 11.

$$\begin{aligned} B3 &= \underline{\mathbf{B}}\times 16 + \underline{\mathbf{3}}\times 1 \\ &= \underline{\mathbf{11}}\times 16 + \underline{\mathbf{3}}\times 1 \\ &= 176 + 3 \\ &= 179 \end{aligned}$$

Example 9: Write the hexadecimal number 76 as a decimal number. (NOTE: Context tells us 76 is a hexadecimal number.)

The 7 is in the sixteens place and the 6 is in the ones place.

$$\begin{aligned} 76 &= \underline{\mathbf{7}}\times 16 + \underline{\mathbf{6}}\times 1 \\ &= 112 + 6 \\ &= 118 \end{aligned}$$

Example 10: Write the hexadecimal number FF as a decimal number.

FF = **F**×16 + **F**×1

= **15**×16 + **15**×1

= 240 + 15

= 255

Notice in the previous example, FF is the largest two-digit hexadecimal number. The subsequent hex number is 100, which would be equivalent to 256 in a decimal system.

Converting Decimals Numbers to Hexadecimal Numbers

As a reminder, the first few powers of 16 are $16^0 = 1$, $16^1 = 16$, and $16^2 = 256$. If we limit our discussion decimal numbers less than 256, we can also say we have limited our hex numbers to two digits.

To convert a decimal number to a hex number we will rely on our old friend from grade school, long division. More specifically, we will rely on long division with remainders. If we divide the decimal number by 16, the quotient (the number on top of the scaffold) will be the number of sixteens, and the remainder will be the number of ones.

Example 11: Write 54 as a hexadecimal number.

How many 16s are in 54?

54 divided by 16 is 3, with a remainder of 6. In other words,…

54 = **3**×16 + **6**×1

= 36_{16} (Remember, the subscript, $_{16}$ indicates the 36 is a hexadecimal number.)

Example 12: Write 194 as a hex number.

194 divided by 16 is 12, with a remainder of 2.

194 = **12**×16 + **2**×1

“12” is not a character in a hex system. We need to use “C” instead.

194 = $C2_{16}$

What is the point behind working with hexadecimal numbers? Admittedly, if you do not work with programming or building computers, you may never actually *see* a hexadecimal number again, but they are very prevalent. Every computer, printer, router and modem on the Internet has a physical location, called a **MAC address** (Media Access Control address). When data is transmitted between those devices, it is directed to a specific MAC address, and that address is comprised of a six-pair set of hexadecimal numbers, such as 00-C0-95-EC-B7-93.

Another common use of hexadecimal numbers is in the colors you find on web pages. For example, if you see yellow text on a page, the underling code would contain a line indicating color=”#FFFF00” with FFFF00 being the hexadecimal representation for the color yellow.

If you plan on a career in electronics, programming, or a related field, you will delve much deeper into the world of binary and hexadecimal numbers. For this course, however, we will limit our discussion to this introduction.

Using Your Calculator

Some (but not all!) scientific calculators are programmed to perform arithmetic tasks using alternative bases. To see if your calculator does so, consult the owner’s manual or look for the keys or functions labeled with DEC, HEX, BIN and OCT (OCT is base 8).

If you calculator possesses this talent, it will not only perform basic arithmetic computations, but it will also convert between the bases when switching modes. For example, if, in the DEC mode, if you enter 25 into the calculator and then switch the calculator to BIN, the number will change to 11001. Then, if you change the base to HEX, the number will change to 19. That confirms $25_{10} = 11001_2 = 19_{16}$.

As binary, hex, and octal numbers are quite prevalent in programming, there are also a plethora of websites and phone apps that will do the conversions quickly and easily. A simple search for “number base converter” will yield tons of options. If we do choose to utilize an electronic tool, we should still learn the concepts laid out earlier in this material. A basic understanding of the different bases and place values will help spot simple mistakes and unnecessary embarrassments.

Section 1.2 Exercises

For Exercises #1-8, write the binary numbers as decimal numbers.

1. 11

2. 110

3. 1100

4. 1001

5. 10011

6. 11101

7. 111100

8. 1010100

For Exercises #9-16, write the decimal numbers as binary numbers.

9. 5

10. 13

11. 24

12. 56

13. 66

14. 78

15. 101

16. 125

For Exercises #17-24, write the hexadecimal numbers as decimal numbers

17. 32

18. 66

19. A3

20. D4

21. 5C

22. BB

23. CF

24. E9

For Exercises #25-32, write the decimal numbers as hexadecimal numbers.

25. 32

26. 11

27. 59

28. 188

29. 120

30. 132

31. 201

32. 250

Section 1.2 Exercise Solutions

1. 11 = 1×2 + 1×1 = 2 + 1 = 3
2. 110 = 1×4 + 1×2 + 0×1 = 4 + 2 + 0 = 6
3. 1100 = 1×8 + 1×4 + 0×2 + 0×1 = 8 + 4 + 0 + 0 = 12
4. 1001 = 1×8 + 0×4 + 0×2 + 1×1 = 8 + 0 + 0 + 1 = 9
5. 10011 = 1×16 + 0×8 + 0×4 + 1×2 + 1×1 = 16 + 0 + 0 + 2 + 1 = 19
6. 11101 = 1×16 + 1×8 + 1×4 + 0×2 + 1×1 = 16 + 8 + 4 + 0 + 1 = 29
7. 111100 = 1×32 + 1×16 + 1×8 + 1×4 + 0×2 + 0×1 = 32 + 16 + 8 + 4 + 0 + 0 = 60
8. 1010100 = 1 ×64 + 0×32 + 1×16 + 0×8 + 1×4 + 0×2 + 0×1 = 64 + 0 + 16 + 0 + 4 + 0 + 0 = 84
9. 5 = 4 + 0 + 1 = 101
10. 13 = 8 + 4 + 0 + 1 = 1101
11. 24 = 16 + 8 + 0 + 0 + 0 = 11000
12. 56 = 32 + 16 + 8 + 0 + 0 + 0 = 111000
13. 66 = 64 + 0 + 0 + 0 + 0 + 2 + 0 = 1000010
14. 78 = 64 + 0 + 0 + 8 + 4 + 2 + 0 = 1001110
15. 101 = 64 + 32 + 0 + 0 + 4 + 0 + 1 = 1100101
16. 125 = 64 + 32 + 16 + 8 + 4 + 0 + 1 = 1111101
17. 32 = 3×16 + 2×1 = 48 + 2 = 50
18. 66 = 6×16 + 6×1 = 96 + 6 = 102
19. A3 = 10×16 + 3×1 = 160 + 3 = 163
20. D4 = 13×16 + 4×1 = 208 + 4 = 212
21. 5C = 5×16 + 12×1 = 80 + 12 = 92
22. BB = 11×16 + 11×1 = 176 + 11 = 187
23. CF = 12×16 + 15×1 = 192 + 15 = 207
24. E9 = 14×16 + 9×1 = 224 + 9 = 233

25. 32 → 32÷16 = 2, Rem 0 → 20

26. 11 → 11÷16 = 0, Rem 11 → 0B (or just B)

27. 59 → 59÷16 = 3, Rem 11 → 3B

28. 188 → 188÷16 = 11, Rem 12 → BC

29. 120 → 120÷16 = 7, Rem 8 → 78

30. 132 → 132÷16 = 8, Rem 4 → 84

31. 201 → 201÷16 = 12, Rem 9 → C9

32. 250 → 250÷16 = 15, Rem 10 → FA

1.3: EXPRESSIONS & EQUATIONS

Variables, Terms & Expressions

A **variable** is a symbol that stands for a number. Although a letter of the alphabet is usually used for a variable, we are not limited to those symbols. We could use a £, a ♣, a λ, or even a ☺. Since those symbols are not readily available on a keyboard, we will stick with letters. Also, in math, the symbols we use are just that – symbols. That means, since "A" and "a" are different characters, they are also different symbols. Even though they are both versions of the same letter, when we use them as symbols, they are as different as "A" and "B."

Think of the characters Δ and δ. Would you call them the same? Δ is the capital Greek letter delta, and δ is how we write the lowercase delta. In math, we would not think of them as Greek letters; we use them as symbols. The letters in the English alphabet need to be treated the same way.

A **term** is the product of numbers and/or variables. In a term, the numeric part is called the **coefficient**. **Like terms** are terms with identical variable parts, including exponents. A strictly numeric term with no variable part is called a constant.

An **expression** is one or more terms, joined by addition or subtraction. Examples of expressions are x, $3xy^2$, -14, and 4a + 2. An expression DOES NOT contain an equal sign. Expressions should be simplified by combining like terms – by combining only the numeric parts. When combining like terms, the variables do not change, only the coefficients do. If the variable parts are different, the terms are not like terms and cannot be combined.

Simplifying Expressions

Think of two boxes of fruit. The first one holds 5 apples and 4 pears, and the second box contains 3 apples and 7 pears. If we combined all of the fruit together, there are 8 apples and 11 pears. Combining like terms is done the same way: (5a + 4p) + (3a + 7p) = 8a + 11p.

Example 1: Simplify the following expression.

3x + 5x – 2

Combine the terms with variables, but leave the constant alone.

3x + 5x – 2 = 8x – 2

When we simplify expressions with a lot of different terms, ONLY like terms can be combined. If we have one cabinet with 5 plates and 6 cups, and combine it with a second cabinet with 4 cups and 7 plates, the result is 12 plates and 10 cups. Notice the order of the items in that sentence. It would be equally correct to say there are 10 cups and 12 plates. When we are putting things together, the addition can be done in any order.

Example 2: Simplify the following expression.

$14a + b – 2a + 4b - 7$

Combine terms containing the terms with the variable a separately from the terms with the b.

$14a + b – 2a + 4b – 7 = 14a – 2a + b + 4b – 7 = 12a + 5b - 7$

In an expression, it is important to notice there is no = sign in the statement of the problem. We are not trying to find a numeric value that "solves" an expression. Expressions can be simplified, but not solved.

Equations and Solutions

An **equation** is a mathematical statement that says two expressions are equal. A **solution** is a value of the variable that makes the equation a true statement.

Example 3: Is $x = 8$ a solution to the equation $x + 6 = 13$? Why or why not?

No. If we substitute the value of 8 for x we get $8 + 6 = 13$, which simplifies to $14 = 13$, which is not true. Thus, $x = 8$ is not a solution to the stated equation.

Example 4: Is $x = 8$ a solution to the equation $x^2 - 7x + 5 = 13$? Why or why not?

Yes. If we substitute the value of 8 for x we get $(8)^2 – 7(8) + 5 = 13$, which simplifies to $64 - 56 + 5 = 13$, or $13 = 13$, which is a true statement. Thus, $x = 8$ is a solution to the stated equation.

The Addition Property of Equality

When we have an expression, our goal is to simplify it. When presented with an equation, our goal is to find the solution. The presence of the equal sign states the equation is balanced. This balance can be maintained and the variable isolated (to reveal the solution) by using the **Addition Property of Equality**. This property states: If the same quantity is added to both sides of an equation, the solution remains the same.

Addition Property of Equality

If the same quantity is added to both sides of an equation, the solution remains the same.

Since any quantity can be added to both sides (as long as it's the same on both sides), we will choose a quantity that allows us to isolate the variable.

Try not to think of this as adding the same quantity to both sides of the equation *at the same time*. Instead, think of adding the needed quantity to the side with the variable in order to isolate that variable term. *Then* perform an identical action to the other side so that the equation remains balanced.

Think of a balance scale – an instrument with two trays. We rarely put items in both trays at the same time. To use it, we put an item in one tray, *and then* add the same weight to the other side to bring things into balance. The two trays are balanced if and only if the weights on each tray are equal.

Example 5: Solve the following equation.

$x - 8 = 24$

In order to isolate the x, we need to eliminate the "- 8." To do that, we will add 8 to the left side of the equation. Then, in order to keep the equation balanced, we also need to add 8 to the right side of the equation. Then simplify each side.

$x - 8 = 24$

$x - 8 \underline{\mathbf{+ 8}} = 24 \underline{\mathbf{+ 8}}$

$x = 32$

The nice thing about solving an equation is we do not need anyone to tell us if we are right or wrong; we can check for ourselves. Remember, a solution must make the original statement true. If we substitute the value $x = 32$ in the original equation, we will find it is, indeed, a solution.

The equation solving process always has us isolating the variable. If something is being added to the variable, we undo that action by subtracting the same amount. Likewise, if something is being subtracted from the variable (as in the previous example) we undo the subtraction by adding that amount to the side with the variable. Then preform the same action to the other side of the equation.

Keep in mind, it doesn't matter which side of the equation has the variable. Even though we may feel more comfortable when the variable is on the left side of the equation, just like when using the balance scales, we can still get the solution when the variable is on the right.

Example 6: Solve the following equation.

43 = 17 + c

In order to isolate the c, we need to eliminate the "17." To do that, we will subtract 17 from the right side of the equation. Then, in order to keep the equation balanced, we also need to subtract 17 from the left side of the equation. Then simplify each side.

43 = 17 + c

43 **- 17** = 17 + c **- 17**

26 = c

Example 7: Solve the following equation.

-23 = 9 + m

In order to isolate the m, we need to eliminate the "9." To do that, we will subtract 9 (or we can add -9) from the right side of the equation. Then, in order to keep the equation balanced, we also need to subtract 9 from the left side of the equation. Then simplify each side.

-23 = 9 + m

-23 **- 9** = 9 + m **- 9**

-32 = m

In the previous example, some a common arithmetic mistake happens with -23 – 9. Some students mistakenly view it as 23 – 9 = 14, and then tack on the – sign to get the incorrect answer of -14. To help with addition and subtraction involving negative numbers, think of your checking account. If you balance is -$23 and you take out $9, is your new balance -$14? If so, there may need to be a conversation between you and your bank. If your balance is already negative and you take money out of your account, your balance better get *more* negative.

The Multiplication Property of Equality

Another property used is the **Multiplication Property of Equality**. This property states: If both sides of an equation are multiplied by the same non-zero quantity, the solution remains the same.

Multiplication Property of Equality

If both sides of an equation are multiplied by the same non-zero quantity, the solution remains the same.

In the practical application of this property, many people like to *divide* both sides of the equation by the same non-zero quantity. We will demonstrate it by multiplying both sides with fractions.

Example 8: Solve the following equation.

$4x = 36$

In order to isolate the x, we need to eliminate the "4." To do that, we will multiply the left side by the fraction 1/4. Then, in order to keep the equation balanced, we also need to do the same thing to the right side of the equation.

$4x = 36$

$(1/4)(4x) = (1/4)(36)$

The 4s on the left cancel. On the right, we have 36/4, which reduces to 9. So,

$x = 9$

Just like before, a solution must make the original statement true. So, if we substitute the value $x = 9$ into the original equation, we will find it is, indeed, a solution.

Be careful! Always ask yourself, "How is the number linked to the variable?" Then note that, when using these properties to solve equations, addition will undo subtraction, while division (or multiplication by a fraction) will undo multiplication.

COMMON MISTAKE

A common mistake is to attempt to undo multiplication with subtraction.

For example, when asked to solve $4x = 36$, some students will subtract 4 from both sides.

Remember, since the 4 and the x are linked by multiplication, we must use division to undo that link. Alternatively, we could multiply by 1/4 to make that 4 cancel.

Then do the same things to the other side of the equation.

Example 9: Solve the following equation.

-3y = 12

The -3 and the y are linked by multiplication. To undo that link and make the -3 cancel, we multiply the left side of the equation by -1/3. If we just multiplied by 1/3, the 3s would cancel, but the negative sign would still be there. Remember, a negative times a negative is positive. By multiplying by -1/3, the entire -3 is cancelled.

-3y = 12

(-1/3)(-3y) = (-1/3)(12)

y = -12/3 = -4

In the previous example, we multiplied by -1/3 because 1/(-3) looks a little awkward. Moving ahead, we will make note of the fact that 1/(-a) is the same as -1/a. In other words, as long as the "-" sign is there, we will be OK. That means we could have multiplied 1/(-3) and got to the same answer.

Next, it is worth paying attention to the "non-zero" part of the property. What happens when we multiply something by zero? Everything becomes 0. That means, since 0(4*x*) = 0, we would actually *lose* the variable. When trying to solve an equation, that doesn't do us any good. Thus, we stay away from 0.

Using the Addition and Multiplication Properties Together

More often than not, an equation contains both addition and multiplication, meaning we need to apply *both* the Addition Property and the Multiplication Property in order to isolate the variable. Once an equation is simplified to the form of ax + b = c, we first use the Addition Property to isolate the term containing the variable. Then as the last step, we multiply both sides of the equation by the reciprocal of the coefficient of the variable term.

Whenever we solve an equation, we should ALWAYS check our solution.

Is y = -4 a solution to -3y = 12? Substituting a -4 for y, we have (-3)(-4) = 12, which is a true statement. That proves y = -4 is a solution to the given equation.

Example 10: Solve the following equation.

$3h - 8 = 7$

First, we need to isolate the 3h by eliminating the "- 8" from the left side. To do that, we add 8 to the left side. Then, to keep the equation balanced, we add 8 to the right side.

$3h - 8 = 7$

$3h - 8 + 8 = 7 + 8$

$3h = 15$

Since there is no more addition (or subtraction) left, we turn to the Multiplication Property. In order to isolate the h, we need to get rid of the coefficient of 3. To do that, multiply the 3h by 1/3 to make the 3s cancel. Then, to keep the equation balanced, we have to multiply the right side by 1/3.

$(1/3)(3h) = (1/3)(15)$

$h = 5$

Finally, check the solution.

$3(5) - 8 = 7$

$15 - 8 = 7$

$7 = 7$

Since 7 = 7 is a true statement, our solution of h = 5 is correct.

Eliminating Mistakes

It is very difficult to locate our own mistakes. Furthermore, those mistakes will not stop happening until we acknowledge the types of mistakes we are prone to making. Most people make the same types of mistakes over and over. Once we step back and recognize the specific types of mistakes that tend to haunt us, we can focus our search for them and become more efficient in correcting them.

Mistakes are part of the natural learning process. Do not regret making them; profit by them. Pay attention to the mistakes you make and consciously look for them in the future.

For example, if you have made the mistake of subtracting away a numeric coefficient in the past, then you are likely to make the same mistake again. If an answer check reveals a mistake has been made, the FIRST kind of mistake for which you should look is that common subtraction error.

Using Your Calculator

When solving equations, a calculator can help with the arithmetic computations, but it will not provide guidance in the problem-solving process. If you are prone to making mistakes in addition, subtraction, multiplication and division, it would be in your best interests to keep your calculator within an arm's reach. Do not, however, become so blinded by the process so that you lose sight of something obvious. If we are faced with the equation $4x = 20$, stop and think. Is it better to blindly divide both sides of the equation by 4, or can you simply ask yourself "4 times what number will give me 20?"

Before we get to the exercises, let's look at a couple more examples.

Example 11: Solve the following equation.

$26 = 5w - 9$

To isolate the w term, add 9 to the right side to eliminate the "- 9." Then add 9 to the left side to keep the equation balanced.

$26 = 5w - 9$

$26 + 9 = 5w - 9 + 9$

$35 = 5w$

Now multiply the right side by 1/5 to make the coefficient cancel. Balance the equation by doing the same to the left side.

$(1/5)(35) = (1/5)(5w)$

$7 = w$

Check:

$26 = 5(7) - 9$

$26 = 35 - 9$

$26 = 26$, which is true. Thus, $w = 7$ is correct.

Example 12: Solve the following equation.

$13 – 6q = 67$

The 13 is positive. To eliminate it, we need to add "-13" (which is the same as subtracting 13) to the left. Then, to keep the equation balanced, add "-13" to the right side. Don't lose the "-" sign in front of the 6q.

$13 – 6q = 67$

$13 – 13 – 6q = 67 – 13$

$-6q = 54$

For the next step, we multiply the left side by 1/(-6). Balance the equation by doing the same to the right side.

$(-1/6)(-6q) = (-1/6)(54)$

$q = (-54)/(6) = -9$

Check:

$13 - 6(-9) = 67$

$13 - (-54) = 67$

$13 + 54 = 67$

$67 = 67$, which is true. Thus, $q = -9$ is correct.

Section 1.3 Exercises

For Exercises #1-11, simplify each expression by combining like terms.

1. $4y + 5y$

2. $23m - 16m$

3. $14x - x$

4. $3q + 5q + 15r - 3r$

5. $32x - 7 - 14x + 10$

6. $12a + 3b - 7a + 2b - 7$

7. $14z + 5y$

8. $11x - 11$

9. $2a + 3 - 4a + b - 11$

10. $(4a + 4p) + (2a - 7p)$

11. $(15m + 4n) + (3m + 9n) - 8n + 11m$

12. Is $x = 2$ a solution to $3x - 6$? Why or why not?

13. Is $y = 2$ a solution to $3x = 6$? Why or why not?

14. Is $a = -3$ a solution to $3a - 6 = -3$? Why or why not?

15. Is $c = 5$ a solution to $3c - 6 = 9$? Why or why not?

For Exercises #16-23, indicate the value that needs to be added to both sides of the equation in order to isolate the variable, and then determine the solution to the equation.

16. $x - 5 = 12$

17. $m - 19 = 7$

18. $y - 4 = 39$

19. $x + 15 = 12$

20. $x + 7 = -9$

21. $a + 13 = 12$

22. $x + (-4) = 10$

23. $m + (-2) = 18$

For Exercises #24-31, indicate the fraction that needs to be multiplied by both sides of the equation by in order to isolate the variable, then determine the solution to the equation.

24. $3m = 39$

25. $5x = 35$

26. $-4y = 12$

27. $-6m = 66$

28. $7x = -28$

29. $32 = 8x$

30. $-9 = -a$

31. $9y = 0$

For Exercises #32-40, solve the indicated equations.

32. $2x - 8 = 26$

33. $2y + 8 = 26$

34. $3m + 2 = -19$

35. $5a - 7 = 30$

36. $6x + 7 = 22$

37. $-7x - 8 = 27$

38. $9 + 2x = 35$

39. $12 = 4x - 28$

40. $-11 = 7 - 5y$

Section 1.3 Exercise Solutions

1. 9y

2. 7m

3. 13x
   NOTE: A common mistake is to subtract the x away and get 14. Think of the x as an object. If we have 14 objects and take away 1 object, we are left with 13 objects.

4. 8q + 12r

5. 18x + 3

6. 5a + 5b – 7

7. 14z + 5y
   NOTE: There are no like terms, so it cannot be simplified. Don't leave it blank, though. If we cannot simplify it, simply write the answer as it stands.

8. 11x – 11
   NOTE: Just like the previous one, since they are not like terms, it cannot be simplified any further.

9. -2a + b – 8
   NOTE: Be careful with the negative signs.

10. 6a – 3p
    NOTE: Since the parentheses are separated by a + sign, they have no impact in the solution process.

11. 29m + 5n
    NOTE: Similar to #8, the parentheses are inconsequential.

12. No. 3x – 6 is not an equation, so it cannot be "solved."

13. No. x = 2 would be a solution, but the stated value is for y.

14. No. 3(-3) – 6 = -3 is false.

15. Yes. 3(5) – 6 = 9 is true.

16. Add +5 to both sides. x = 17

17. Add 19 to both sides. m = 26

18. Add +4 to both sides. y = 43

19. Add -15 to both sides. x = -3

20. Add -7 to both sides. x = -16

21. Add -13 to both sides. a = -1

22. Add +4 to both sides. x = 14

23. Add +2 to both sides. m = 20

24. Multiply both sides by 1/3. m = 13

25. Multiply both sides by 1/5. x = 7

26. Multiply both sides by -1/4. y = -3

27. Multiply both sides by -1/6. m = -11

28. Multiply both sides by 1/7. x = -4

29. Multiply both sides by 1/8. x = 4

30. Multiply both sides by -1. a = 9

31. Multiply both sides by 1/9. y = 0

32. Add 8 to both sides to get 2x = 34. Then multiply both sides by 1/2 to get x = 17.

33. Subtract 8 from both sides to get 2y = 18. Then multiply both sides by 1/2 to get y = 9.

34. Subtract 2 from both sides to get 3m = -21. Then multiply both sides by 1/3 to get m = -7.

35. Add 7 from both sides to get 5a = 37, Then multiply both sides by 1/5 to get a = 7.4.

36. Subtract 7 from both sides to get 6x = 15. Then multiply both sides by 1/6 to get a = 2.5.

37. Add 8 to both sides to get -7x = 35. Then multiply both sides by -1/7 to get x = -5.

38. Subtract 9 from both sides to get 2x = 26. Then multiply both sides by 1/2 to get x = 13.

39. Add 28 to both sides to get 40 = 4x. Then multiply both sides by 1/4 to get x = 10.

40. Subtract 7 to both sides to get -18 = -5y. Then multiply both sides by -1/5 to get y = 3.6.

1.4: FAIR DIVISION

Fair Division is the concept of dividing something among two or more people in such a way that each person finds his/her share to be fair. There are a number of ways to achieve fair division, and some of them will be investigated in this section. Before we begin, however, we must clarify one very important point. Divisions do not have to be equal; they just need to be fair in the eye of the beholder. Each party should get what they deem to be their fair share, with minimal concern for the other parties. Also, the parties involved in the division are expected to be reasonable. Otherwise, an independent arbitrator must oversee the procedure.

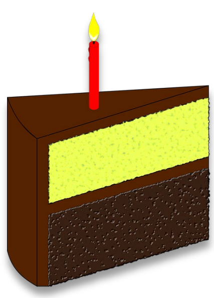
Fair Division DOES NOT Mean Equal Division

Discrete items, such as cash or a box full of CDs, can usually be split fairly. Some **continuous items**, such as pets and cars cannot be cut up, as they would lose value or become totally worthless (not to mention a bit messy). Other continuous items, like pizza, can be split up without losing value. Many times, continuous items that cannot be split are liquidated into discrete items, such as money, to complete a division.

The Divider-Chooser Method

The **Divider-Chooser Method** is a practical and simple approach to having two people share an object that can be cut into pieces. You may already be familiar with this process.

Let's say two children, a brother and sister, want to share a piece of cake. While this could result in a terrible argument, there is a simple, yet fair way to divide that piece of cake.



1. First, flip a coin to determine which child will be Player 1. The other will be Player 2.

2. Next, Player 1 gets to cut the piece of cake into two parts, wherever he/she wishes.

3. Then, Player 2 gets to choose the piece of cake he/she would like to have, leaving the other piece for Player 1.

In this system, one player gets to divide the cake, and the other player gets to choose which piece to take.

Equal Division of an Item that Can be Liquidated

If we are trying to divide an item among two or more people, one simple way is to **liquidate** (sell) the item and then split the proceeds. If, however, one or more of the involved parties wants the item instead of cash, we need to figure out how one person can get the item and then pay the others for their shares of it.

If college roommates Dan and Dave purchased a car together and are now graduating, logically they are both entitled to half the car. So, if both of them agreed the car was worth $8000, instead of selling it, Dan could take the car and give Dave $4000, which would make the division equal and fair.

If there were four roommates wanting to split a car, each is entitled to one fourth of the value of the car. So, for the same $8000 car, Dan could take it and pay $2000 to each of the other three. This works well when everyone involved agrees on the value of the item that needs to be divided. The situation quickly becomes more complicated when sentimental value creeps into the situation.

Remember, fair division does not mean equal division. When the involved parties think the item in question is worth a different amount, the amount they are entitled to receive changes. If there are two people, each person is entitled to half of *what they think* the item is worth. When three people are involved, each is entitled to one third of *what they think* the item is worth.

The Method of Sealed Bids

Developed by Polish mathematician **Bronislaw Knaster** in 1945, the **Knaster Inheritance Procedure**, also called the **Method of Sealed Bids**, is completed using the following procedure:

1. Each person writes down the highest amount they would be willing to pay for each item, without knowing what the other parties bid for the same item. Assume there are no ties in the bids.

2. The highest bidder for each item will win it, and the other people will be compensated with money, paid to them by the person who won the item. Bidders, however, are not compensated directly. Instead, if there are n bidders, the winner will place (n - 1)/n×(the winning bid) into a kitty. For example, if there are 4 bidders, the winner places 3/4 of his bid into the kitty.

3. The non-winners will then take 1/nth of their individual bids from the kitty. For example, if three people are involved in the division, each person who did not win is entitled to 1/3 of the amount of the amount he/she bid for that item.

4. After each non-winner has taken the corresponding amount from the kitty, the remaining **residual funds** in the kitty are then divided equally among each bidder, including the one who won the item.

Multiplying by a Fraction

Focusing on the second step, pay attention to the amount the winner needs to place into the kitty. If there are four bidders, the winner is only entitled to 1/4 of his bid, and the remaining 3/4 of his bid will be used to cover the compensations for the others. If there are 10 bidders the winner is entitled to 1/10 of his bid, and the remaining 9/10 of his bid goes into the kitty to cover the other’s shares.

When multiplying by a fraction, multiply the dollar amount by the numerator (top number) and divide by the denominator (bottom number).

Example 1: 4 people can claim joint ownership of an item and intend to divide it by the Method of Sealed Bids. If the winner offers a bid of $8,000, how much will that bidder need to put into a kitty to compensate the other bidders?

Since each person is entitled to 1/4 of their bid, that means the winner must put 3/4 of his bid into the kitty.

(3/4)($8000) = $6000

Now let's look at an example of the whole procedure. When a bidder takes money out of their pocket and puts it into the kitty, we will list this a negative amount. Alternatively, money gained will be positive.

Example 2: Jenny and Amy are no longer going to be roommates, but they have to decide who will get to keep their dog. Jenny bids $300 for the dog, and Amy bids $400 for it. With the higher bid, Amy gets the dog. How much will she have to pay Jenny to make the division fair?

Since this division involves two people, each of them is entitled to 1/2 of the value of their bid. To start the process, Amy will have to put 1/2 of her bid, $200, into the kitty.

Then, Jenny takes 1/2 of her bid, $150, from the kitty.

This leaves a residual of $50 in the kitty, which is split equally between the two people. So, each woman will take $25 from the kitty.

Putting it together,

Jenny' Share: $150 (original money from the kitty) + $25 (1/2 of the residual amount) = $175

Amy's Share: Dog -$200 + $25 = Dog - $175

That means, to be fair, Amy gets the dog and pays $175 to Jenny.

In the previous example, notice how the $175 changing hands is the same for both parties. If our work had shown +$200 for Jenny, but -$150 for Amy, the discrepancy in the cash would be an indication of a mistake.

In a sealed bids division, in the end, all the money must be accounted for.

Example 3: Audrey, Brian and Cindy have inherited a sculpture and want to divide it. Audrey bids \$4200, Brian bids \$6000, and Cindy bids \$5100 for it. Use the Method of Sealed Bids to determine the fair division.

Brian is the high bidder, so he will get the sculpture. There are three people involved in this division, so he will put 2/3 of his \$6000 bid into a kitty. 2/3×\$6000 = \$4000

Audrey bid \$4200 on it, so she is entitled to 1/3 of \$4200, which is \$1400. Likewise, Cindy is entitled to 1/3 of \$5100, which is \$1700.

The kitty started with \$4000. After Audrey and Cindy each remove their shares, a residual \$900 remains in the kitty. That \$900 is divided equally among all three people, so each of them receives \$300.

That makes the final division,

Audrey's Share: \$1400 + \$300 = \$1700

Brian's Share: Sculpture - \$4000 + \$300 = Sculpture - \$3700

Cindy's Share: \$1700 + \$300 = \$2000

In other words, Brian gets the sculpture, and gives \$1700 to Audrey and \$2000 to Cindy. Audrey gets less because she valued the sculpture at a lower amount than Cindy.

Example 4: Diana and Bruce are splitting up and want to divide a dinette set they purchased together. Diana thinks the set is worth \$800, while Bruce thinks it is worth \$500. Use the Method of Sealed Bids to determine the fair division.

Diana gets the dinette set and puts \$400 into the kitty.

Bruce takes \$250 from the kitty.

The residual \$150 in the kitty is split equally between the happy couple. Each of them gets \$75.

That makes the division,

Diana's Share: Dinette Set - \$400 + \$75 = -\$325

Bruce's Share: \$250 + \$75 = \$325

That means Diana gets the dinette set and gives Bruce \$325.

Example 5: Apollo, Athena and Artemis all claim joint ownership of a magic helmet used by their father, Zeus. Using the Method of Sealed Bids, Apollo bids \$6000 for the helmet and Athena bids \$3000 for it, while Artemis only bids \$900. Apollo will win the helmet and must pay cash to his siblings. How much does he have to pay to each of them to make the division fair?

Since each of the three children has a rightful claim to 1/3 of the helmet, in order to cover the thirds that will be claimed by his two siblings, Apollo must put 2/3 of his bid into a kitty. Thus, he will start the process by putting \$4000 into the kitty.

Athena is entitled to 1/3 of her bid, which is \$1000, and Artemis is entitled to 1/3 of his bid, which is \$300. When those shares are taken from the kitty, there will be a residual of \$2700. When that residual is split equally among the three siblings, each of them will get \$900.

In the end, Athena will get \$1000 from the kitty and then \$900 more when the residual is split, making a total of \$1900. Likewise, Artemis will take \$300 from the kitty and then get \$900 of the residual, giving him a total of \$1200. For the honor of taking his father's helmet, Apollo will put \$4000 into the kitty, but get \$900 of it back. So, of the \$3100 Apollo has to give his siblings, \$1900 goes to Athena and \$1200 will be given to Artemis.

In the Method of Sealed Bids, since the highest bidder is placing money into the kitty, there will always be a little bit of money left after the other people have taken their shares. For example, when there are two bidders, half of the highest bid is always more than half of the lower bid. When that little bit of extra money is divided, the end result is the non-winning bidders end up with slightly more than 1/nth of their bid. And, the winning bidder will receive the item by paying less than the amount that was actually bid. Thus, by either being awarded the item or by receiving slightly more than a fair share, the Method Sealed Bids often leaves each person satisfied.

Let's get a little more complicated and examine the division where there are multiple items under consideration. If we have more than one item, we compute the division for each item separately, and then add the results together at the end.

Example 6: Eric and Fran need to divide a collection of stuffed animals and an antique chair. Eric thinks the stuffed animals are worth \$400 and the chair is worth \$1200. Fran thinks the stuffed animals are worth \$300, while the chair is worth \$1500. Use the Method of Sealed Bids to determine the fair division.

Compute the division of each item independently and add the results together at the end.

Example 6 Continued:

For the stuffed animals,

Eric gets the stuffed animals and puts $200 into a kitty.

Fran takes $150 from the kitty.

The residual $50 in the kitty is split equally, so each of them gets $25.

For the chair,

Fran takes the chair and puts $750 into a kitty.

Eric takes $600 from the kitty.

They split the residual $150, which is $75 each.

Putting it all together,

Fran's Share: $150 + $50 + Chair - $750 + $75 = Chair - $500

Eric's Share: Stuffed Animals - $200 + $25 + $600 + $75 = Stuffed Animals + $500

That means Fran gets the chair, and Eric gets the stuffed animals and $500 from Fran.

Section 1.4 Exercises

1. Two friends want to share the last cookie. Describe how it may be divided using the Divider-Chooser method.

2. 3 people can claim joint ownership of an item and intend to divide it by the Method of Sealed Bids. If the winner offers a bid of $9,000, how much will that bidder need to put into a kitty to compensate the other bidders?

3. 6 people can claim joint ownership of an item and intend to divide it by the Method of Sealed Bids. If the winner offers a bid of $9,000, how much will that bidder need to put into a kitty to compensate the other bidders?

4. 2 people can claim joint ownership of an item and intend to divide it by the Method of Sealed Bids. If the winner offers a bid of $19,000, how much will that bidder need to put into a kitty to compensate the other bidders?

5. 10 people can claim joint ownership of an item and intend to divide it by the Method of Sealed Bids. If the winner offers a bid of $4,500, how much will that bidder need to put into a kitty to compensate the other bidders?

6. Ron and Joe have inherited a car and need to divide it. Ron thinks the car is worth $11,400 and Joe bids $10,200 for it. Using the Method of Sealed Bids, how should they do the division?

7. Laura and Ryan have inherited a boat and need to divide it. Laura thinks the boat is worth \$9,300 and Ryan bids \$8,400 for it. Using the Method of Sealed Bids, how should they do the division?

8. Donnie and Marie have decided to go their separate ways, and each can currently claim joint ownership of a specially designed custom stage set. Donnie says the set is worth \$10,000, while Marie thinks it is worth \$12,000. As the higher bidder, Marie gets the set. Using the Method of Sealed Bids, how much must she pay to Donnie to make the division fair?

9. Peter, Edward and Rose have decided they no longer want to be business partners and have agreed to use the Method of Sealed Bids to allow for one of them to buy the business from the others. Pete bids \$90,000 for the business, Edward bids \$120,000, and Rose says the business is worth \$195,000. How much must Rose pay to Peter and Edward to keep the business for herself and make the division fair?

10. The school year has ended, and three roommates, Willie, Mickey and Duke need to divide an old couch. Willie thinks the couch is worth \$90, Mickey says it is worth \$120, and Duke bids \$78 for it. Using the Method of Sealed Bids, how should they do the division?

11. The school year has ended, and three roommates, Willie, Mickey and Duke need to divide TV. Willie thinks the TV is worth \$330, Mickey says it is worth \$414, and Duke bids \$480 for it. Using the Method of Sealed Bids, how should they do the division?

12. A singing quartet has broken up, and Soprano, Alto, Tenor and Bass need to divide their tour bus. Soprano bids \$52,000, Alto thinks it is worth \$48,000, Tenor bids \$64,000, and Bass bids \$68,000. Using the Method of Sealed Bids, how should they complete the division?

13. A singing quartet has broken up, and Soprano, Alto, Tenor and Bass need to divide their plane. Soprano bids \$76,000, Alto thinks it is worth \$80,000, Tenor bids \$68,000, and Bass bids \$72,000. Using the Method of Sealed Bids, how should they complete the division?

14. In the process of a divorce, Jeff and Charlotte are going to use the Method of Sealed Bids to determine who will get their car. Jeff bids $24,000 and Charlotte bids $21,500, so Jeff will get the car. How much will he have to pay Charlotte in order to keep the division fair?

15. John and Ken need to divide a couch, a set of dishes, and a refrigerator. John thinks the couch is worth $80, the dishes are worth $30, and the refrigerator is worth $60. Ken bids $40 on the couch, $40 on the dishes and $80 on the fridge. Using the Method of Sealed Bids, how will the items be divided?

16. Ruby and Kim need to divide two items: a couch and a television. Ruby thinks the couch is worth $160 and the TV is worth $400. Kim bids $120 on the couch and $300 on the TV. Ruby's bids are higher on both items, so, using the Method of Sealed Bids, how much should she pay Kim to make the division fair?

Section 1.4 Exercise Solutions

1. They should flip a coin to determine the person who will cut the cookie. Then, the other person can select either of the two pieces.

2. (2/3)($9000) = $6000

3. (5/6)($9000) = $7500

4. (1/2)($19,000) = $9,500

5. (9/10)($4500) = $4050

6. Ron gets the car and puts $5700 into a kitty. Joe takes $5100 out of the kitty. They split the residual $600 into $300 each.
   Ron: Car - $5700 + $300 = Car - $5400
   Joe: $5100 + $300 - $5400
   Ron gets the car and has to give $5400 to Joe.

7. Laura gets the boat and puts $4650 into a kitty. Ryan takes $4200 out of the kitty. They split the residual $450 into $225 each.
   Laura: Boat - $4650 + $225 = Boat - $4425
   Ryan: $4200 + $225 = $4425
   Laura gets the boat and has to give $4425 to Ryan.

8. Marie takes the stage set and puts $6000 into a kitty. Donnie takes $5000 out of the kitty. They split the residual $1000 into $500 each.
   Donnie: $5000 + $500 = $5500
   Marie: Stage Set - $6000 + $500 = Stage Set - $5500

9. Rose takes the business and puts 2/3 off her $195,000 bid, which is $130,000, into a kitty. Peter takes $30,000 and Edward takes $40,000 out of the kitty. They split the residual $60,000 three ways, which is $20,000 each.
   Peter: $30,000 + $20,000 = $50,000
   Edward: $40,000 + $20,000 = $60,000
   Rose: Business - $130,000 + $20,000 = Business - $110,000
   Rose pays $50,000 to Peter and $60,000 to Edward.

10. Mickey gets the couch and puts $80 into a kitty. Willie takes $30 and Duke takes $26 out of the kitty. They split the residual $24 into $8 each (remember, there are three people).
    Willie: $30 + $8 = $38
    Mickey: Couch - $80 + $8 = Couch - $72
    Duke: $26 + $8 = $34
    Mickey gets the couch and gives $38 to Willie and $34 to Duke.

11. Duke gets the TV and puts \$320 into a kitty. Willie takes \$110 and Mickey takes \$138 out of the kitty. They split the residual \$72 into \$24 each.
    Willie: \$110 + \$24 = \$134
    Mickey: \$138 + \$24 = \$162
    Duke: TV - \$320 + \$24 = TV - \$296
    Duke gets the TV and gives \$134 to Willie and \$162 to Mickey.

12. Bass gets the bus and puts \$51,000 into a kitty. Soprano takes \$13,000, Alto takes \$12,000, and Tenor takes \$16,000 from the kitty. The residual of \$10,000 is split between the four singers, which is \$2500 each.
    Soprano: \$13,000 + \$2,500 = \$15,500
    Alto: \$12,000 + \$2,500 = \$14,500
    Tenor: \$16,000 + \$2,500 = \$18,500
    Bass: Bus - \$51,000 + \$2,500 = Bus - \$48,500
    Bass gets the bus and gives \$15,500 to Soprano, \$14,500 to Alto, and \$18,500 to Tenor.

13. Alto gets the plane and puts \$60,000 into a kitty. Soprano takes \$19,000, Tenor takes \$17,000, and Bass takes \$18,000 from the kitty. The residual of \$6,000 is split between the four singers, which is \$1500 each.
    Soprano: \$19,000 + \$1,500 = \$20,500
    Alto: Plane - \$60,000 + \$1,500 = Plane - \$58,500
    Tenor: \$17,000 + \$1,500 = \$18,500
    Bass: \$18,000 + \$1,500 = \$19,500
    Alto gets the plane and gives \$20,500 to Soprano, \$18,500 to Tenor, and \$19,500 to Bass.

14. Jeff gets the car and puts \$12,000 into a kitty. Charlotte takes \$10,750 out of the kitty. The residual of \$1250 is split into \$625 each.
    Jeff: Car - \$12,000 + \$625
    Charlotte: \$10,750 + \$625 = \$11,375
    Jeff has to pay \$11,375 to Charlotte.

15. For the couch, John puts \$40 into a kitty, and then Ken takes \$20 out. They split the residual \$20 into \$10 each. For the dishes, Ken puts \$20 into a kitty, and then John takes \$15 out. They split the residual \$5 into \$2.50 each. For the fridge, Ken puts \$40 into a kitty, and then John takes \$30 out. They split the residual \$10 into \$5 each.
    John: Couch - \$40 + \$10 + \$15 + \$2.50 + \$30 + \$5 = Couch + \$22.50
    Ken: \$20 + \$10 + Dishes - \$20 + \$2.50 + Fridge - \$40 + \$5 = Dishes + Fridge - \$22.50
    John gets the couch and \$22.50 from Ken. Ken gets the dishes and the refrigerator, and pays \$22.50 to John.

16. For the couch, Ruby puts \$80 into a kitty, and then Kim takes \$60 out. They split the residual \$20 into \$10 each. For the TV, Ruby puts \$200 into a kitty, and then Kim takes \$150 out. They split the residual \$50 into \$25 each.
    Ruby: Couch - \$80 + \$10 + TV - \$200 + \$25 = Couch + TV - \$245
    Kim: \$60 + \$10 + \$150 + \$25 = \$245
    Ruby takes the couch and TV, and needs to pay Kim \$245.

1.5: PROBLEM SOLVING

From the Wizard of Oz

(Scarecrow)
I could wile away the hours
Conferrin' with the flowers
Consultin' with the rain
And my head I'd be scratchin'
While my thoughts were busy hatchin'
If I only had a brain

I'd unravel any riddle
For any individ'le
In trouble or in pain

(Dorothy)
With the thoughts you'd be thinkin'
You could be another Lincoln
If you only had a brain

Do you hate word problems? Why? Don't just say, “Because they are hard.” A problem-solving expert once stated, "Quite simply, students cannot solve word problems reliably because they are presented with inconsistent models of problem solving that contradict the logical processes they have learned in other courses and in everyday life."

The National Council of Teachers of Mathematics (NCTM) takes the following stand:

"Problem solving - which includes the ways in which problems are represented, the meanings of the language of mathematics and the ways in which one conjectures and reasons - must be central to schooling so that students can explore, create, accommodate to changed conditions, and actively create new knowledge over the course of their lives."

Unfortunately, students are rarely focused on the problem-solving process; they just want the answers. Additionally, many teachers and textbooks blow right past the "thinking" part of the problem and place too much importance on the answer. The most essential part of the whole education process - learning how to think critically - is left out of many of the examples presented to the student. The disastrous result is students being left with the perception that the problems worked in math class are somehow different from those presented in real life or other subject matter courses such as physics, chemistry, and business. They aren’t.

The Problem-Solving Process

Many students struggle with word problems because they don't believe reading is a part of a math class, and thus, they don't take the time to read and understand the problems.

In his book, *How to Solve It*, **George Polya** laid out a step-by-step process for problem solving:

1. Understand the Problem

2. Devise a Plan

3. Carry Out the Plan

4. Look Back

A fifth step - MAKE SURE YOU ANSWER THE QUESTION - is also very important. The bottom line is - Use your common sense when working with word problems.

For a more descriptive list of Polya's process, consider the following:

Step 1: Understand the Problem
- Do we understand all the words?
- Can we restate the problem in our own words?
- Do we know what is given?
- Do we know what the goal is?
- Is there enough information?
- Is there extraneous information?
- Is the problem similar to another problem we have solved?
- Draw a picture of the problem statement, if applicable.

Step 2: Devise a Plan. There are many ways to solve problems. Essentially, a strategy is defined as an artful means to an end. Can one of the following strategies be used?
- Guess and Test: Yes, guessing is a legitimate strategy for solving a problem.
- Look for a Pattern: Often, patterns can point to a formula.
- Use a Variable and Translate the English Phrases into Algebraic Expressions and/or Equations.
- Make a List: Many times, a list will reveal a pattern.
- Solve a Simpler Problem: Often, if we cannot solve a problem, there is an easier problem that lies at the root of our troubles. Finding and solving that easier problem may lead to clues that help solve the bigger one.
- Draw a Picture of How to Solve the Problem: Yes, a picture can be worth 1000 words.
- Work Backwards: Works great if we know the answer and need to find the process.

Step 3: Carry Out the Plan
- Implement the strategy or strategies we have chosen until the problem is solved or a new course of action is suggested.
- Give ourselves a reasonable amount of time in which to solve the problem.
- Don't be afraid of starting over. Often, a fresh start and a new strategy will lead to success.

Step 4: Look Back
- Is our solution correct?
- Does our answer satisfy the statement of the problem?
- Can we see an easier solution?
- Can we use this solution to solve other problems?

Step 5: Answer the Question

- ➢ In written or verbal responses, we should always answer the asked question in a complete sentence.

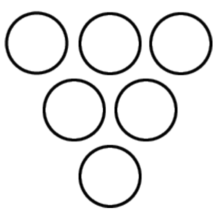
The following activities will help us get a better understanding of strategies mentioned in Step 2. Each strategy comes with a list of clues and a practice problem, and is followed by the solution. Be sure to solve the problem using the suggested strategy, and, in some cases, a hint may be provided. You may want to cover the solution while you work on each problem.

Problem Solving Strategy: Guess and Test

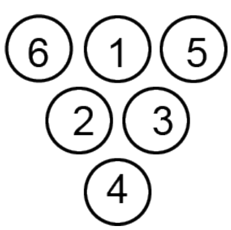
Clues: The **Guess and Test** strategy may be appropriate when:

- ➢ There are a limited number of possible answers.
- ➢ We want to gain a better understanding of the problem.
- ➢ We have a good idea of what the answer is.
- ➢ We can systematically try possible answers.
- ➢ There is no other obvious strategy to try.

Example 1: Place the digits 1, 2, 3, 4, 5, and 6 in the circles below so that the sum of the three numbers on each side of the triangle is 12. Hint: There are a limited number of possible configurations. Start plugging in the numbers and you may actually stumble across the correct answer!



Multiple solutions are possible. Here's one:



Problem Solving Strategy: Look for a Pattern

Clues: The **Look for a Pattern** strategy may be appropriate when:

- A list of data is given.
- Listing special cases helps you deal with complex problems.
- We are asked to make a prediction or generalization.

Example 2: What is the sum of the first n consecutive odd numbers?

1 = 1

1 + 3 = ______

1 + 3 + 5 = ______

1 + 3 + 5 + 7 = ______

1 + 3 + 5 + 7 + 9 = ______

1 + 3 + 5 + 7 + 9 + 11 = ______

Continue this until you see a pattern.

If you think you have the solution, predict the sum of the first 15 odd numbers. That is, without doing the addition, what is the following sum?

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 = ______

$1 = 1 = 1^2$
$1 + 3 = 4 = 2^2$
$1 + 3 + 5 = 9 = 3^2$
$1 + 3 + 5 + 7 = 16 = 4^2$
$1 + 3 + 5 + 7 + 9 = 25 = 5^2$
$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

The sum of the first n consecutive odd numbers is n^2. Thus, the sum of the first 15 odd numbers is $15^2 = 225$.

Problem Solving Strategy: Use a Variable

Clues: The **Use a Variable** strategy may be useful when:

- A phrase similar to "for any number" is present or implied.
- A problem suggests an equation.
- A proof or a general solution is required.
- There is a large number of cases
- A proof is asked in a problem involving numbers.
- We are trying to develop a general formula.

Example 3: What is the greatest number that divides the sum of any three consecutive whole numbers, without leaving a remainder?

Understand the problem: The whole numbers are 0, 1, 2, 3, 4,… Consecutive whole numbers differ by 1.

Devise a Plan: If n represents the first whole number, what are the next two? How do we indicate the sum of such numbers?

Carry Out the Plan: n + (n+1) + (n+2) = ? Can you finish this one?

Look Back, Check Our Work, and Answer the Question

If n is the first number, then any three consecutive whole numbers are n, n+1, and n+2.

The sum of those three numbers is: n + (n+1) + (n+2) = 3n + 3 = 3(n + 1).

So, for any three consecutive whole numbers, the sum will always be 3(n + 1). That means you can divide that result by 3 without leaving a remainder.

Problem Solving Strategy: Make a List

Clues: The **Make a List** strategy may be appropriate when:
- The information can easily be organized and presented.
- Data can be easily generated.
- Asked "in how many ways" something can be done.

Example 4: In how many ways can the letters A, B, and C be arranged?

Listing all the possibilities, ABC, ACB, BAC, BCA, CAB, & CBA, we see there are six possible arrangements.

Problem Solving Strategy: Solve a Simpler Problem

Mathematician George Polya once said, “If you cannot solve a problem, there is a simpler problem you cannot solve. Find it.” After all, sometimes we need to figure out what is wrong before we can figure out what is right.

Clues: The **Solve a Simpler Problem** strategy may be appropriate when:
- The problem involves complicated computations.
- The problem involves very large or small numbers.
- We are asked to find the sum of a series of numbers.
- A direct solution is too complex.

Example 5: There are 20 people at a meeting. If each person shakes hands with every person (except himself) in the room only once, how many handshakes will there be? Hint: What if there were 2 people? 3 people? 4 people? …

With 2 people there is 1 handshake. If a third person walks into the room, he shakes two hands, making a total of three handshakes.

Continuing this observation, we see:

2 People = 1 Handshake
3 People = 3 Handshakes
4 People = 6 Handshakes
5 People = 10 Handshakes

When a new person, n, enters the room, he shakes n-1 hands (he does not shake hands with himself), making the total number of handshakes for n people equal to the sum of the first n-1 whole numbers. For example, for 5 people, the number of handshakes is 4+3+2+1=10. The sum of the first n-1 whole numbers is n(n-1)/2, so, for 20 people there will be (20)(19)/2 = 190 handshakes.

Problem Solving Strategy: Draw a Picture

Clues: The **Draw a Picture** strategy may be appropriate when:

- A physical situation is involved.
- Geometric figures or measurements are involved.
- A visual representation of the problem is possible.

Example 6: We can make one square with four toothpicks. Show how we can make two squares with seven toothpicks (breaking toothpicks is not allowed), three squares with 10 toothpicks, and five squares with 12 toothpicks. (Draw some pictures!)

With 7, one side of each square shares a side. That is, draw two squares that are side-by-side, and share one side. Likewise, with 10 toothpicks we can line up three squares in a row, with the center square sharing sides with each of the adjacent ones.

For 12 toothpicks, first make a large square with 8 toothpicks, using two toothpicks on each side. Then use the remaining four toothpicks to make a cross through the center, dividing the big square into 4 small ones. Thus, we have four small squares, and one large one.

Problem Solving Strategy: Work Backwards

Clues: The **Work Backwards** strategy may be appropriate when:

- We know the answer and need to find the process.

Example 7: A merchant with a basket of oranges sells half of them to the first person and then gives him one more for good measure. He then sells half the remaining oranges to the second person and gives him an extra orange for good measure. A third person buys exactly half the remaining oranges and the vendor gives him one more for good measure. Finally, the merchant eats the last orange. How many oranges were originally in the basket?

As the strategy indicates, work backwards. The merchant ate the last orange. Before that, he gave one free orange to the third customer, meaning, just before that, there had to be two oranges in the basket. The third customer bought half the oranges, and that purchase left two oranges. Thus, before the purchase, there had to be four oranges.

Continuing to work backwards, the second customer was given a free orange. That means, before the gift orange was given, there had to be five oranges in the basket. That second customer bought half the oranges, and five were left. That means, before the purchase, there had to be 10 oranges in the basket.

Before the first customer was given a free one, there had to be 11 oranges in the basket. Which, similar to before, means there were 22 oranges in the basket before the first customer made his purchase.

Now that we know there were 22 oranges in the basket, it is always a good idea to verify the solution by working through the problem going forward. Half of 22 is 11. Give away 1 to get 10. Half of 10 is 5. Give away 1 to get 4. Half of 4 is 2. Give away 1 and enjoy the last orange.

Using Your Calculator

Be careful with your calculator, especially when you are asked to find an arithmetical pattern. It is really easy for a calculation to result in a number with too many digits for a calculator to display. This will result in the calculator rounding the answer to a value that fits its display. And, remember, if we are not given a place value, it is improper to round to one out of convenience.

Example 8: Find the squares of 6, 66, 666, and 6666. Then, establish the pattern and use that pattern to predict the squares of 66,666 and 666,666 and 666,666,666?

Example 8 Continued

Using a calculator, $6^2 = 36$, $66^2 = 4356$, $666^2 = 443,556$ and $6666^2 = 44,435,556$

The pattern is the number of 4s and 5s in the answer is one less than the number of 6s in the number you were squaring. Separate the 4s and 5s with a 3, and end the number with a 6.

So,

$66,666^2$ will have four 4s and four 5s, separated with a 3, and ending with a 6. That means $66,666^2 = 4,444,355,556$.

$666,666^2$ will have five 4s and five 5s, separated with a 3, and ending with a 6. That means $666,666^2 = 444,443,555,556$.

$666,666,666^2$ will have eight 4s and eight 5s, separated with a 3, and ending with a 6. That means $666,666,666^2 = 444,444,443,555,555,556$.

If you used your calculator to compute $666,666^2$ you may have gotten the wrong answer! Most scientific calculators can only display eight or ten digits. So, to display the twelve-digit 444,443,555,556 the calculator converts the number to scientific notation and rounds it to an eight- or ten-digit number: $4.444435556 \times 10^{11}$. Even worse, many calculators do not display the "$\times$ 10" of the notation, leaving just a gap between the last digit and the exponent: 4.444435556 11. Furthermore, in some calculators an "E" may be displayed in place of the "$\times$ 10": 4.4444435556E11. Neither of those answers has the requisite number of 5s.

Likewise, on a normal ten-digit scientific calculator, $666,666,666^2$ is displayed as 4.4444444436 17 (or 4.4444444436E17) – missing the 5s altogether! If you converted the displayed answer back to standard form by moving the decimal point 17 places to the right, you would have 444,444,443,600,000,000, and that value is over 44 million away from the correct answer of 444,444,443,555,555,556.

Graphing calculators can display larger numbers, but, alas, they too, have a limit. In general, it would be wise to follow the instructions and actually establish and use the indicated pattern. ☺

Section 1.5 Exercises

1. A student started out poorly on his first mathematics test. However, he doubled his score on each of the next two tests. The third test grade was 96. What was the student's average (mean) test grade?

2. A large drum filled with water is to be drained using a small opening at the top. One 1-inch diameter hose or three 1/2-inch hoses can be used to siphon out the water. Would it be faster to use the one 1-inch hose or the three 1/2-inch hoses? Why?

3. The cost of a car increases by 20% and then decreases by 20%. Is the resulting price of the car greater than, less than, or equal to the original price of the car? Why?

4. A new high school graduate receives two job offers: Company A offers a starting salary of $15,000 a year with a $600 raise every 6 months, while Company B offers $15,000 a year with a $1200 raise every 12 months. Which offer will provide the most income?

5. Fill in the three blanks using some combination of the symbols +, -, ×, and ÷ to make a true statement of equality.

   7 ___ 7 ___ (7 ___ 7) = 13

6. While visiting a friend's home, I saw kittens and children playing in the backyard. Counting heads, I got 18. Counting feet, I got 60. How many kittens and how many children were in the backyard?

7. Determine the pattern and then generate the next two iterations for the following.

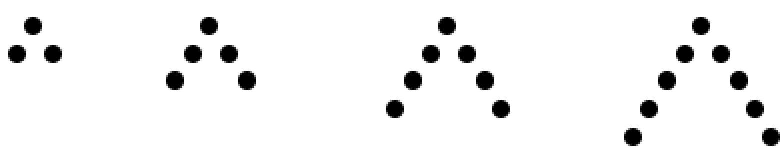
   $37 \times 3 = 111$
   $37 \times 6 = 222$
   $37 \times 9 = 333$

8. Place the numbers 2, 4, 6, 8, 10, 12, 14, 16, and 18 in the squares below so that the sum of the numbers in every column, row, and diagonals is equal to 30.

9. Peter, Paul, and Mary are three sports professionals. One is a tennis player, one is a golfer, and one is a skier. They live in three adjacent houses on City View Drive. From the information below determine who the professional skier is.
   - Mary does not play tennis.
   - Peter does not play golf.
   - The golfer and the skier live beside each other.
   - Three years ago, Paul broke his leg skiing and has not tried it since.
   - Mary lives in the last house.
   - The golfer and the tennis player share a common backyard swimming pool.

10. The time in New York City is one hour ahead of the time in St. Louis, and three hours ahead of the time in Las Vegas. If a flight left NYC at 9 AM, stopped in St. Louis for 50 minutes, and then arrived in Las Vegas at 1:35 PM, how long was the plane actually flying?

11. If the following pattern is continued, how many dots will be in the hundredth figure?



12. A man who has a garden 10 meters square (10 m by 10 m) wishes to know how many posts will be required to enclose his land. If the posts are placed exactly 1 m apart, how many are needed? Disregard the thickness of the posts.

13. Determine the pattern and then generate the next two iterations for the following.

$9^2 = 81$
$99^2 = 9801$
$999^2 = 998,001$
$9999^2 = 99,980,001$
$99,999^2 = 9,999,800,001$

14. Cindy was given her allowance on Monday. On Tuesday she spent \$1.50 on candy. On Wednesday, Cindy found \$1.00 on the ground. If Cindy now has \$2, how much was her allowance?

15. John, Paul, and George uncovered a treasure chest containing some diamonds. They buried half of the diamonds and divided the remaining diamonds evenly among themselves. John received 20 diamonds. How many diamonds were in the treasure chest when they found it?

16. Joe walked from his home to Susan's in 20 minutes. Then, together, it took Joe and Susan 25 minutes to walk from Susan's house to school. If they arrived at school at 7:15 AM, what time did Joe leave his home?

17. Gym lockers are to be numbered from 1 to 99 using metal numbers to be glued onto each locker. How many 7s are needed?

18. At the beginning of the bake sale, Mary set aside 4 pies for herself. Karen then bought half of the remaining pies, and 5 more pies were sold during the sale. When the sale was over, there were 3 pies remaining. How many pies were there before the bake sale started?

19. Calculators were purchased at $65 per dozen and sold at $20 for three calculators. Find the profit for selling six dozen calculators.

20. A backyard fair charged \$1.00 admission for adults and \$0.50 for children. The fair made \$25 and sold 38 tickets. How many adult tickets were sold?

21. Mike wants to cut a log into 10 pieces. How many cuts are necessary?

22. A person has 10 coins consisting of dimes and quarters. If the person has a total of \$1.90, find the number of quarters.

23. Use your calculator to determine the squares of 75, 175, 275, 375 and 475. Use this pattern to predict the square of 575 and 675.

24. 111,111,111/12,345,679 = 9
    222,222,222/12,345,679 = 18
    333,333,333/12,345,679 = 27
    444,444,444/12,345,679 = 36
    What is 888,888,888/12,345,679? (Use the pattern, not your calculator!)

25. $11^2 = 121$
    $111^2 = 12{,}321$
    $1{,}111^2 = 1{,}234{,}321$
    $11{,}111^2 = 123{,}454{,}321$
    What is $111{,}111^2$? (Use the pattern, not your calculator!)

Section 1.5 Exercise Solutions

1. Technique: Algebra. If x is the first score, 2x is the second, and 4x (2x doubled) is the third. Thus, 4x=96 and, therefore, x=24. His three test scores are 24, 48, and 96.
   The mean score is (24+48+96)/3 = 56.

2. Technique: Draw a Picture - . Of course, if we really want the algebra, we need to use the formula for the area of a circle, which is A = πr2. When we compute the area of the two circles and we will find the 1-in hose (with an area of $\pi/4$) will do the job faster than three of the 1/2-in hoses (with an area of $3\pi/16$).

3. Technique: Use an Example. Let's say the car was originally \$10,000. If the price increases by 20%, the new price is \$10,000 + 0.20(\$10,000) = \$12,000. If that price is then decreased by 20%, you must subtract 20% of the \$12,000, not the \$10,000. Since 0.20(\$12,000) = \$2400, and \$12,000-\$2400=\$9600, the new price is less than the original price.

4. Technique: Make a list. If you look at the monthly salaries for the first year, you will find the person will be making a higher salary for 6 months with Company A.

5. Technique: Trial and Error. One way to do it is 7+7-(7÷7).

6. Technique: Algebra. Assuming children have one head and two legs, and kittens also have one head, but four legs... Using C for the number of children and K for the number of kittens, let C+K=18 and 2C+4K=60. Solving the first equation for C and substituting into the second equation, you get 2(18-K)+4K=60. Solve to find K=12. Thus, there are 12 kittens and 6 children.

7. Technique: Look for a Pattern. Notice $37 \times 3n = nnn$. So, $37 \times 12 = 444$, and $37 \times 15 = 555$.

8. Technique: Look for a Pattern. Notice that 2+18+10=30, 4+16+10=30, 6+14+10=30, and 8+12+10=30. The 10 gets used in each of the four sums, and the only place that crosses four directions (both diagonals, across the middle, and down the center) is the center square. Put 8, 6 and 16 in the top row, put 18, 10 and 2 in the middle row, and put 4, 14 and 12 in the bottom row.

9. Technique: Make a List and use the process of elimination. First, you can eliminate tennis for Mary. Then, since the golfer shares a backyard with the other two, the golfer must be in the middle house. Since Mary lives in the last house, she is not the golfer, and, hence, must be the skier.

10. Technique: Solve a Simpler Problem. That is, note that when the plane lands in Vegas, the time is 1:35 PM, which is 4:35 PM in NYC. Thus, the time from start to finish 7 hours and 35 minutes (9 AM to 4:35 PM). Deduct the 50 minutes of layover time in STL, and you will find the plane was in the air for 6 hours and 45 minutes.

11. Technique: Look for a Pattern. The nth figure has 2n+1 dots. Therefore, the 100th figure has 201 dots.

12. Technique: Draw a Picture. 40 posts are needed.

13. Technique: Look for a Pattern. The pattern is to start the number with consecutive 9s that number one less than the number of 9s in the original number. Then, follow the nines with an 8, and then include as many consecutive 0s as there are nines to start the number. Then, end the number with a single 1. Thus, for 999,9992 we start with five 9s, then an 8, then five 0s, and end with a 1: 999,998,000,001. Likewise, 9,999,9992 = 99,999,980,000,001

14. Technique: Work Backwards. She ends with $2. Take away the $1 she found, and she had $1 before the found money. Add in the $1.50 she spent, and we find her allowance was $2.50.

15. Technique; Work Backwards. They each got the same amount of diamonds, so, together they got a total of 60 diamonds. 60 is half of the remaining, so there had to be 120 diamonds in the chest when they found it.

16. Technique: Work Backwards. They arrived at school at 7:15. 25 minutes earlier was 6:50 AM. 20 minutes before that would be 6:30 AM.

17. Technique: Make a List. Twenty 7's are needed.

18. Technique: Work Backwards. 20 pies.

19. Technique: Simplify the Problem. Think of them in dozens, not individuals. $90 profit.

20. Technique: Algebra. 12 adult tickets were sold.

21. Technique: Draw a Picture. 9 cuts are needed.

22. Technique: Guess and Test. 4 dimes and 6 quarters.

23. Technique: Be sure to use the pattern and not your calculator. 5752 = 330,625, 6752 = 455,625

24. Technique: Be sure to use the pattern and not your calculator. 72

25. Technique: Be sure to use the pattern and not your calculator. 12,345,654,321

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CHAPTER 2: GEOMETRY & MEASUREMENT

LET NO MAN IGNORANT OF GEOMETRY ENTER HERE

That phrase was inscribed over the door to **Plato's Academy** in Athens. The ancient Greeks were the pioneers of **geometry**. Before them, the Egyptians and the Babylonians used many geometric principles in practical applications, but it was the Greeks who studied its philosophical properties. Traditionally, in most high school geometry classes, students are subjected to numerous two-column proofs, which are not the most fun or interesting things in the world. While there is merit in learning those proofs and techniques, the beauty and extreme usefulness of geometry gets lost. Whether we are looking at the circular wheels on a bicycle, or the straight lines and right angles of walls, ceilings and doorways, without a basic understanding of geometry our world would be a very different place.

CHAPTER OUTLINE AND OBJECTIVES

Section 2.1: The Metric System & Dimensional Analysis

A. Understand the relative sizes of metric measures.
B. Be able to perform conversions between different measures using unit fractions.
C. Be able to perform a conversion with multiple units using dimensional analysis.

Section 2.2: Polygons, Circles & Solids

A. Be able to identify various polygons and their properties.
B. Be able to understand concepts involving circles, including radius, diameter and pi.
C. Be able to identify different 3-dimensional figures and their properties.

Section 2.3: Perimeter, Area & Volume

A. Be able to find the perimeter and area of different polygons.
B. Be able to find the circumference and area of circles.
C. Be able to find the perimeter and area of composite figures.
D. Be able to find the surface area and volume of various solids.

Section 2.4: Ratios, Rates & Proportions

A. Be able to understand ratios and rates.
B. Be able to understand speed, mileage & unit pricing.
C. Be able to solve proportions.
D. Be able to work with similar triangles.

Section 2.5: Right Triangles, Distance & Slope

A. Be able to compute square roots.
B. Be able to use the Pythagorean theorem to solve a right triangle.
C. Be able to find distances using the Pythagorean theorem.
D. Be able determine slope in various applications.

2.1: THE METRIC SYSTEM & DIMENSIONAL ANALYSIS

According to Wikipedia, the **United States** is the only industrialized country that has not adopted the **metric system** as its official system of measurement. We actually had a good chance to do it in the 1970's when Richard Nixon was president, but a government study showed it would have been "prohibitively expensive." Mechanics would have had to buy new tools, cars would have had to change their speedometers and odometers, speed limit and distance signs had to be changed, information printed on packages of food needed to be changed, hospitals had to switch equipment from standard imperial measuring tools to metric ones, and the list went on, and on, and on, and on… Oddly enough, if we take another look at that list, we will see most (if not all) of those entities are currently using metric measures.

So, if it would no longer be prohibitively expensive why doesn't the United States go metric now? If we took a poll, we would be told, "The metric system is too hard." Really? Then, if we asked the same people if they would rather work with decimals or fractions, nearly all of them would quickly say "decimals because fractions are too hard."

Additionally, consider the following two questions, and ask yourself which one is easier to answer.

- What measurement is between 17 millimeters and 19 millimeters?
- What measurement is between 9/16 of an inch and 5/8 of an inch?

Nearly everyone will agree that decimals are much easier to work with than fractions. By the way, in case you didn't realize it, the metric system is already the primary – if not exclusive – system of measure in nearly all medical settings. As a courtesy, many of those medical settings convert the metric measures of centimeters and kilograms to feet and pounds when reporting information to the patients and relatives.

Most Americans do not know their height in centimeters and their weight in kilograms, and they don't really want to learn those measures, either. Somebody once said, "The only ones that like a change are wet babies." That person was probably a politician afraid of not being re-elected. Since it would literally take an act of Congress to officially make the change to the metric system, if a politician voted to make someone's life inconvenient (by forcing them to learn metric measures!), he or she would likely lose a bid for re-election. In reality, if Congress mandated an instant 100% adoption of the metric system, there would be a rough period of adjustment, but we would be far better off in just a few years. As different countries around the world made the change to the metric system, they often reported a 2-3 year "adjustment" period. Simply put, Americans are stubborn.

Historical Measurements

Before the metric system, many of the standards for measurements came from England. The original version of this system (also called to **Imperial** or **English System**) had some seemingly odd measurements. Since few people before the Industrial Revolutions of the 18th and 19th Centuries cared about exact measurements, there were many discrepancies in the basic definitions. For example, the foot was literally the length of the **foot** of the King of England (hence the name "foot"). The **yard** was the distance from the tip of the nose to the end of an outstretched arm, the **inch** was the length of three barley corns laid end to end, a **fathom** was the length of a full arm span, and an **acre** was the amount of land a horse could plow in one day. When these units became standardized, the foot was defined by the length of a prototype metal bar, the inch was defined as 1/12 of a foot, the yard as the length of three feet, and so on.

Then Along Came the French

In the late 1700's, the French government devised a system of measure that was easily portable, convertible, and interrelated. The new system was based entirely on a single measurement, the **meter**. They defined the meter as one ten-millionth of the distance from the equator to the North Pole along the line of longitude passing through Paris. Then, they divided it according to multiples of 10, giving each multiple a corresponding prefix.

Here are some of the more common prefixes.

Prefix (abbreviation)	Multiple or Fraction	Example
mega- (M-)	$10^6 = 1,000,000$	megameter (Mm)
kilo- (k-)	$10^3 = 1000$	kilogram (kg)
hecto- (h-)	$10^2 = 100$	hectoliter (hL)
deka- or deca- (da-)	$10^1 = 10$	decameter (dam)
	Base Unit: meter (m), gram (g), liter (L)	
deci- (d-)	$10^{-1} = 1/10 = 0.1$	decimeter (dm)
centi- (c-)	$10^{-2} = 1/100 = 0.01$	centimeter (cm)
milli- (m-)	$10^{-3} = 1/1000 = 0.001$	milligram (mg)
micro- (µ- or mc-)	$10^{-6} = 1/1,000,000 = 0.000001$	microgram (µg or mcg)

IMPORTANT: Although unit names are ordinary words, the unit symbols are case-sensitive. For example, mm stands for millimeter (one-thousandth of a meter), but Mm is the megameter (one million meters). Next, the units do not have singular and plural forms. That is, there is no “s” at the end for more than one unit. Finally, there is no period after any of these abbreviations, unless it falls at the end of a sentence.

To expand their system into weights and volumes, the French went on to define a **cubic centimeter**, which is simply a cube that measures 1 cm on each side. If we fill the cube with water (at sea level), we have a **milliliter**. If we weigh that milliliter of water, we have a **gram**. Apart from the decimal nature of the metric system, a distinct advantage of the system is that connection between length, volume and weight (mass). Essentially, 1 cm^3 = 1 mL = 1 g. Thus, with a single milliliter of water, one can produce the entire metric system. Then, once we have the basic measures of length, volume and mass, we can multiply by the appropriate factor to produce any other measurement in the entire system. That’s it. Portable. Convertible. Interrelated.

So, just how big are these metric units?

The biggest hurdle in adopting the metric system is the general lack of understanding of the relative size of the various units. Since most of us grew up measuring things in inches, feet, miles, pounds, quarts and gallons, we intuitively understand the relative size of each of those units. When we watch a basketball game, we understand the players are usually over six feet tall. We print pictures in the dimensions of 4x6, 5x7, and 8x10, and we know those measures are stated in inches. A grown man weighs around 200 pounds, and when we pick up a large stone, we understand it weighs about five pounds. We have all held a gallon of milk and even used simple measuring cups. When we literally experience these measures, they just make sense.

Unfortunately, most of us have not chosen to relate those same everyday life experiences with the metric system. If we truly want to understand the metric system, that understanding must be connected to our lives in a meaningful way.

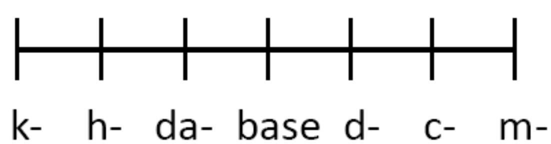
As a frame of reference, here are some basic comparisons:

- A meter is slightly longer than a yard.
- Your little finger is about one centimeter wide.
- A dime is one millimeter thick.
- A kilometer is a little more than a half-mile.
- A dollar bill weighs about one gram.
- A nickel weighs exactly 5 grams.
- A kilogram is about the weight of a large tub of butter (2.2 pounds).
- A liter is a little more than a quart.
- A milliliter is about a drop.
- A kiloliter is over 260 gallons.

Converting Between Metric Units

Converting between units in the metric system is easy - once we learn those prefixes. Until then, consider using a metric converter like the following.

To use the metric converter, locate the prefix we are given, count how many places we must move in order to get to the desired prefix, and then, move the decimal point the same number of places and in the same direction as we did on the converter.



Example 1: Convert 3.2 dm to mm.

Locate dm on the converter, and then notice we need to move two places to the right to get to mm. Thus, we will also move the decimal point in 3.2 two places to the right.

3.2 dm = 320 mm

Remember, when converting between metric measures, all we are doing it moving the decimal point. Then, realize, when the unit gets larger, to maintain equality, the number must get smaller. Likewise, when the unit gets smaller, the number must get larger.

Unit Fractions

A **unit fraction** is a fraction that is equivalent to 1. Unit fractions are extremely useful when we want to convert from one unit of measure to another. To create a unit fraction, simply make a fraction with two equal measures that have different units. For example, since 6ft = 2yd, both $\frac{6\ ft}{2\ yd}$ and $\frac{2\ yd}{6\ ft}$ are unit fractions.

Dimensional Analysis

Dimensional analysis is just that – an analysis of the dimensions. To perform a unit conversion, we analyze a conversion based on the existing units of measure, multiply by the unit fraction corresponding to the desired conversion, cancel the unwanted units, and multiply and/or divide the numbers based on their location in the resulting expression. The appropriate unit conversion fraction is a fraction made up of two equivalent measures (like 1 meter and 100 centimeters) AND will make the unwanted unit of measure cancel while leaving the desired unit.

Example 2: Use dimensional analysis to convert 3.4 meters to centimeters.

We are starting with 3.4 m. Since we want an ending unit of cm, we need to multiply by the unit fraction between meters and centimeters.

1 meter = 100 centimeters. Thus, we have two choices for our desired unit fraction:

$\frac{1\ m}{100\ cm}$ or $\frac{100\ cm}{1\ m}$.

If we multiply 3.4 m by the first fraction listed above, the "m" will not cancel. Thus, in order to get the "m" to cancel, we multiply by $\frac{100\ cm}{1\ m}$. To aid in the process of the cancellation, put the 3.4 m over 1:

$\frac{3.4\ m}{1} \times \frac{100\ cm}{1\ m}$

Notice how the m's cancel just like a common factor, and the unit of measure left in the answer is cm. Then, multiply and divide the numeric quantities in the fraction. 3.4×100 = 340.

3.4 m = 340 cm

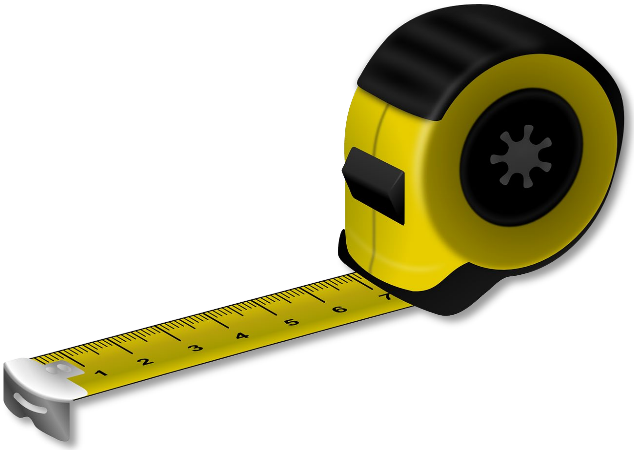
At this point, you may be thinking, "That metric converter is a lot easier." Being a visual device, that may be true, but it is extremely limited. Dimensional analysis, however, is an extremely powerful and flexible tool, and it always works.

Conversion Factors

Before we expand on our presentation of dimensional analysis, let's review some common – and a few not so common - **conversion factors**. Those marked with an asterisk (*) are approximate.

Length:

- 12 inches (in) = 1 foot (ft)
- 3 ft = 1 yard (yd)
- 5280 ft = 1 mile (mi)
- 1 mi = 1.61 km
- 2.54 cm = 1 in
- 1 m = 1.09* yd

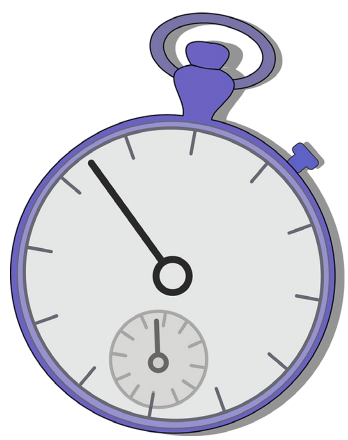


Weight/Mass:

- 16 ounces (oz) = 1 pound (lb)
- 2000 lb = 1 ton (t)
- 1 kg = 2.2* lb
- 1 ton = 0.9* metric ton (mt)
- 1 stone (st) = 14 lb
- 1 oz = 28.4* g

Volume:

- 3 teaspoons (tsp) = 1 tablespoon (tbsp)
- 2 tbsp = 1 fluid ounce (fl oz)
- 8 fl oz = 1 cup (c)
- 2 cup = 1 pint (pt)
- 2 pint = 1 quart (qt)
- 4 qt = 1 gallon (gal)
- 1 tsp = 5* mL
- 1 L = 1.057* qt
- 1 cm^3 = 1 cubic centimeter (cc) = 1 mL
- 1 ft^3 = 7.48* gal



Time:

- 60 seconds (sec) = 1 minute (min)
- 60 min = 1 hour (h or hr)
- 24 hr = 1 day
- 365.25* days = 1 year (yr)
- It is OK to use 365* days = 1 year

When performing a conversion with dimensional analysis create an appropriate unit fraction from the conversion factors listed above.

In the vast wasteland often referred to as the Internet, many, many pages exist offering many, many conversion factors. We'll offer a few words about online converters in a bit. For now, however, let's focus on dimensional analysis and unit fractions.

Example 3: Use dimensional analysis to convert 5 feet to inches.

We are starting with 5 ft. Since we want an ending unit of in, we need to multiply by the unit fraction involving feet and inches.

12 in = 1 ft. Thus, we have two choices for our desired unit fraction: $\frac{12\ in}{1\ ft}$ or $\frac{1\ ft}{12\ in}$

If we multiply 5 ft by the second fraction listed above, the "ft" will not cancel. Thus, in order to get the "ft" to cancel, we need to multiply by $\frac{12\ in}{1\ ft}$. To aid in the process of the cancellation, put the 5 ft over 1.

$$\frac{5\ ft}{1} \times \frac{12\ in}{1\ ft}$$

Notice how the "ft" cancels just like a common factor, and the unit of measure left in the answer "in." Then, multiply the numeric quantities in the fraction.

5 ft = 5×12 in = 60 in

Example 4: Use dimensional analysis to convert 85 ft/sec to miles/hour. Round to the nearest whole number.

We have multiple conversions in this one. We need to convert the feet to miles, and the seconds to hours. Thus, we will need the unit fractions containing 5280 ft = 1 mi, and something for time. We may not know a direct seconds-to-hours conversion, but we can use one that converts seconds to minutes and another that converts minutes to hours.

$$\frac{85\ ft}{1\ sec} \times \frac{1\ mi}{5280\ ft} \times \frac{60\ sec}{1\ min} \times \frac{60\ min}{1\ hr}$$

Cancel the "ft," "sec" and "min," which leaves the units of "mi" in the numerator and "hr" in the denominator. Then multiply and divide the numbers, as indicated.

$$\frac{85 \times 60 \times 60\ mi}{5280\ hr} \approx \frac{57.95\ mi}{hr}$$

To the whole number, 85 ft/sec = 58 mi/hr.

Example 5: Kim is 1,000,000,000 seconds old today. What is her age in years?

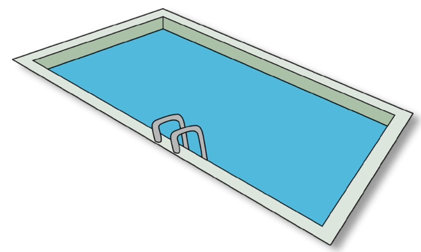
We do not have a seconds-to-years conversion, so we will need to convert the seconds to minutes, the minutes to hours, the hours to days, and the days to years.

$$\frac{1{,}000{,}000{,}000\ sec}{1} \times \frac{1\ min}{60\ sec} \times \frac{1\ hr}{60\ min} \times \frac{1\ day}{24\ hr} \times \frac{1\ yr}{365\ days} \approx 31.70979\ldots$$

A person's age is stated in a whole number of years, **without rounding**.

Thus, Kim is 31 years old.

Example 6: To purchase the appropriate amount of chlorine, Jim needs to know how many gallons of water are in his swimming pool. He measures the length to be 30 feet, the width is 15 feet, and the depth is a uniform 5 feet.



That makes the volume 30 ft × 15 ft × 5 ft = 2250 ft^3. About how many gallons is this?

1 ft^3 = 7.48 gallons. Thus,

$$\frac{2250\ ft^3}{1} \times \frac{7.48\ gal}{1\ ft^3} = 16{,}830 \text{ gal}$$

When indicating the number of gallons of water in a swimming pool, it is customary to round to the nearest thousand. Jim's pool holds about 17,000 gallons of water.

Using Your Calculator

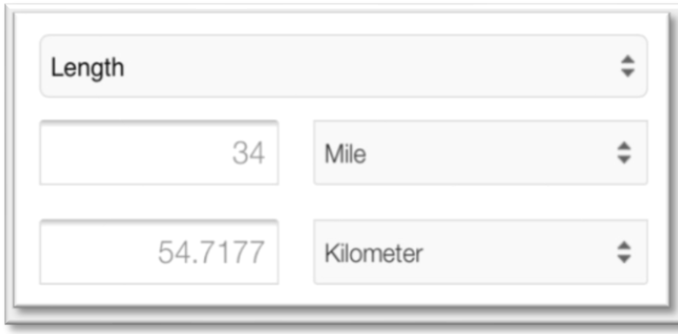
Calculator usage in a dimensional analysis problem is pretty straight-forward. If we are faced with multiple conversions in the same problem, we need to remember to multiply the numbers in the numerators and divide by all the numbers in the denominators.

Using an Online Converter

Dimensional analysis is an extremely useful tool, and something every student should learn. If we are faced with an unfamiliar or complicated conversion, setting up a conversion by analyzing the dimensions will always make things less complicated. In practical situations however, given the ubiquitous nature of computers and cell phones, it's almost silly to not make use of an online converter.

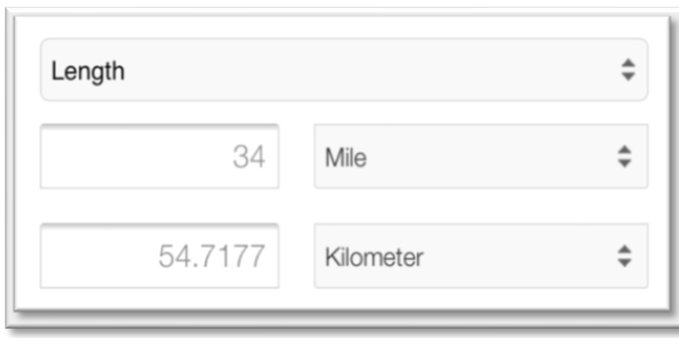
A simple **Google** search for "measure converter" will not only yield millions of results, but it will also show an embedded converter directly within the search results. We don't even need to visit any of the websites.

Using the Google measure converter is really easy. Simply choose the category by clicking the top area, enter the given numeric value and unit, and select the desired unit at the bottom. The converted measure will automatically appear. In fact, the default conversion appearing when we first see the converter is 1 ft = 0.000189394 mi.



Example 7: On your phone or computer, do a Google search for "measure converter" and use the embedded converter to convert 34 miles to kilometers. Round your answer to the nearest whole kilometer.

Using the converter, leave the measure category as Length, enter the 34 and set the given measure to Mile. When we set the desired unit to Kilometer, 54.7177 appears in the bottom left box.



To the nearest whole kilometer, 34 mi = 55 km.

Section 2.1 Exercises

For Exercises #1-9, of the three measurements offered, identify the most realistic measure for the indicated object.

1. Length of a small paper clip	28 mm	28 cm	28 m
2. Distance between Las Vegas and Reno, NV	800 cm	800 m	800 km
3. Height of a building	20 cm	20 m	20 km
4. Height of a 12-year-old boy	148 mm	148 cm	148 m
5. The volume of juice container	4 mL	4 L	4 kL
6. Weight of an apple	100 mg	100 g	100 kg
7. Volume of a tablespoon	15 mL	15 cL	15 L
8. The weight of an eyelash	300 mg	300 g	300 kg
9. The volume of water in a swimming pool	50 mL	50 L	50 kL

For Exercises #10-21, perform the indicated conversions.

10. 5 dm = ______ mm

11. 31 m = ______ cm

12. 0.76 hm = ______ m

13. 7.5 cm = ______ mm

14. 3.06 m = ______ mm

15. 764 m = ______ km

16. 3.22 kg = _______ g

17. 5 mL = ______ L

18. 0.034 L = _______ cL

19. 95 mg = _____ cg

20. 475 cg = _______ g

21. 32 g = _______ mg

22. One of the world's largest slot machine jackpots is in Nevada and it is called Megabucks. Why is the name Megabucks appropriate?

For Exercises #23-38, demonstrate the indicated conversion using dimensional analysis. Round to the nearest tenth, if necessary.

23. 6 cm = ______ in

24. 31 min = ______ sec

25. 0.76 mi = ______ ft

26. 75 qt = ______ pt

27. 168 hr = ______ days

28. 182 lb = ______ st

29. 182 lb = ______ kg

30. 5 mi = ______ yd

31. Dan weighs 65 kg. How many pounds does he weigh?

32. Peyton weighs 32 kg. How many pounds does she weigh?

33. A Smart Car has a curb weight of 1610 lb. How many kg is this?

34. Sydney is 109 cm tall. To the nearest whole number, how many inches tall is she? How tall is she in ft and in?

35. Joe's dentist tells him to use 1.5 tbsp of medicated mouthwash. How many milliliters is this?

36. How many mL are in a pint?

37. How many cups are in a gallon?

38. A sprinter can run 100 yds in 13 seconds. What is his speed in miles per hour?

39. Calvin's horse runs 450 feet in 13 seconds. What is the speed of the horse in miles per hour?

40. A patient is to receive 200 mL of fluid over 1 hour, and the nurse will use tubing rated at 10 drops per mL (gtt/mL) to regulate the IV. Determine the IV flow rate in drops per minute (gtt/min).

41. A patient is to receive 75 mL of fluid over 1 hour, and the nurse will use tubing rated at 20 drops per mL (gtt/mL) to regulate the IV. Determine the IV flow rate in drops per minute (gtt/min).

Section 2.1 Exercise Solutions

1. 28 mm

2. 800 km

3. 20 m

4. 148 cm

5. 4 L

6. 100 g

7. 15 mL

8. 300 mg

9. 50 kL

10. $\frac{5\ dm}{1} \cdot \frac{100\ mm}{1\ dm}$ = 500 mm

11. $\frac{31\ m}{1} \cdot \frac{100\ cm}{1\ m}$ = 3100 cm

12. $\frac{0.76\ hm}{1} \cdot \frac{100\ m}{1\ hm}$ = 76 m

13. $\frac{7.5\ cm}{1} \cdot \frac{10\ mm}{1\ cm}$ = 75 mm

14. $\frac{3.06\ m}{1} \cdot \frac{1000\ mm}{1\ m}$ = 3060 mm

15. $\frac{764\ m}{1} \cdot \frac{1\ km}{1000\ m}$ = 0.764 km

16. $\frac{3.22\ kg}{1} \cdot \frac{1000\ g}{1\ kg}$ = 3220 g

17. $\frac{5\ mL}{1} \cdot \frac{1\ L}{1000\ mL}$ = 0.005 L

18. $\frac{0.34\ L}{1} \cdot \frac{100\ cL}{1\ L}$ = 3.4 cL

19. $\frac{95\ mg}{1} \cdot \frac{1\ cg}{10\ mg}$ = 9.5 cg

20. $\frac{475\ cg}{1} \cdot \frac{1\ g}{100\ cg}$ = 4.75 g

21. $\frac{32\ g}{1} \cdot \frac{1000\ mg}{1\ g}$ = 32,000 mg

22. The prefix mega- stands for million.

23. $\frac{6\ cm}{1} \cdot \frac{1\ in}{2.54\ cm}$ = 2.4 in

24. $\frac{31\ min}{1} \cdot \frac{60\ sec}{1\ min}$ = 1860 sec

25. $\frac{0.76\ mi}{1} \cdot \frac{5280\ ft}{1\ mi}$ = 4012.8 ft

26. $\frac{75\ qt}{1} \cdot \frac{2\ pt}{1\ qt}$ = 150 pt

27. $\frac{168\ hr}{1} \cdot \frac{1\ day}{24\ hr}$ = 7 days

28. $\frac{128\ lb}{1} \cdot \frac{1\ st}{14\ lb}$ = 13 st

29. $\frac{182\ lb}{1} \cdot \frac{1\ kg}{2.2\ lb}$ = 82.7 kg

30. $\frac{5\ mi}{1} \cdot \frac{1760\ yd}{1\ mi}$ = 8800 yd

31. $\frac{65\ kg}{1} \cdot \frac{2.2\ lb}{1\ kg}$ = 143 lb

32. $\frac{32\ kg}{1} \cdot \frac{2.2\ lb}{1\ kg}$ = 70.4 lb

33. $\frac{1610\ lb}{1} \cdot \frac{1\ kg}{2.2\ lb}$ = 731.8 kg

34. $\frac{109\ cm}{1} \cdot \frac{1\ in}{2.54\ cm}$ = 43 in, and 43ft/12 = 3 ft 7 in

35. $\frac{1.5\ tbsp}{1} \cdot \frac{3\ tsp}{1\ tbsp} \cdot \frac{5\ mL}{1\ tsp}$ = 22.5 mL

36. $\frac{1\ pt}{1} \cdot \frac{1\ qt}{2\ pt} \cdot \frac{1\ L}{1.057\ qt} \cdot \frac{1000\ mL}{1\ L}$ = 473.0 mL

37. $\frac{1\ gal}{1} \cdot \frac{4\ qt}{1\ gal} \cdot \frac{2\ pts}{1\ qt} \cdot \frac{2\ cups}{1\ pt}$ = 16 cups

38. $\frac{100\ yds}{13\ sec} \cdot \frac{1\ mi}{1760\ yd} \cdot \frac{60\ sec}{1\ min} \cdot \frac{60\ min}{1\ hr}$ = 15.7 mi/h

39. $\frac{450\ ft}{13\ sec} \cdot \frac{1\ mi}{5280\ ft} \cdot \frac{60\ sec}{1\ min} \cdot \frac{60\ min}{1\ hr}$ = 23.6 mi/h

40. $\frac{200\ mL}{1\ hr} \cdot \frac{10\ gtt}{1\ mL} \cdot \frac{1\ hr}{60\ min}$ = 33.3 gtt/min

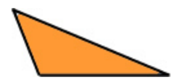
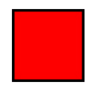
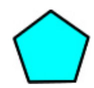
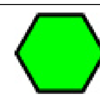
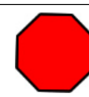
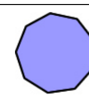
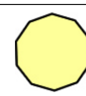
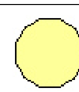
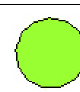
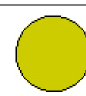
41. $\frac{75\ mL}{1\ hr} \cdot \frac{20\ gtt}{1\ mL} \cdot \frac{1\ hr}{60\ min}$ = 25 gtt/min

2.2: POLYGONS, CIRCLES & SOLIDS

A **polygon** is a plane figure composed of line segments. Additionally, the line segments start and end at the same point, so we consider polygons to be “closed” figures. Even though some may have sides that intersect each other, we will restrict our conversations to simple, non-self-intersecting polygons, which we will classify according to the number of their sides.

Common (and some not-so-common) Polygon Names

The prefix *poly-* means many and the suffix *-gon* means angle. So, translated literally, a polygon is a many-angled figure. Since the number of sides is always equal to the number of angles, we usually categorize polygons by the number of sides. In general, a polygon with n sides can be referred to as an **n-gon**. If, however, we can be more specific, it is wise to do so.

Name	Number of Sides	Picture
Triangle	3	
Quadrilateral	4	
Pentagon	5	
Hexagon	6	
Octagon	8	
Nonagon	9	
Decagon	10	
Dodecagon	12	
Triskaidecagon	13	
Icosagon	20	
Hectogon	100	

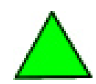
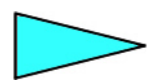
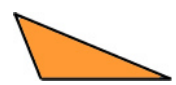
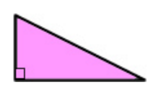
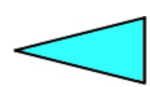
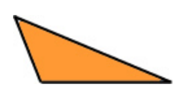
Notice how, as the number of sides of the polygon increases, the figure looks more and more like a circle. At the age of 19, **Carl Friedrich Gauss** proved it was possible to construct a regular **heptadecagon** (17 sides) using only a compass and a straightedge. He was so proud of the proof; he requested the shape be engraved onto his tombstone. When Gauss died, the stonemason refused to perform the complicated task, stating the shape would have been indistinguishable from a circle.

As pointed out earlier, the polygon names are defined by the number of sides. With the exception of triangles and quadrilaterals, all of the names have the suffix -gon. From there, tri- means 3, quad- means 4, penta- means 5, and hexa-, octa-, & nona- mean 6, 8 & 9, respectively. Deca- means 10, so a decagon has ten sides. Can you recognize the prefixes for 12, 13, 20, and 100?

By the way, in the **Roman Lunar Calendar**, October was originally the 8^{th} month, November (Nov- is an alternative prefix for nine) was originally the ninth month, and December was the tenth month. That original Roman calendar consisted of ten months and began with March. **Julius Caesar** eventually reformed the calendar to 12 months and renamed a couple of mid-year months to honor himself and his nephew **Augustus**, whom he later adopted.

Types of Triangles and Their Properties

If we focus on **triangles**, we can see even further classifications.

Triangle	Properties	Picture
Equilateral	All sides are equal, all angles are equal	
Isosceles	At least two sides are equal	
Scalene	No sides are the same length, no angles are equal	
Right	One of the angles measures 90°	
Acute	All angles measure less than 90°	
Obtuse	One angle measures greater than 90°	

You may have noticed the same triangle appearing as an isosceles and an acute triangle (just rotated 180°). This does not mean all isosceles triangles are acute; it just means the pictured isosceles triangle is acute. We could easily have an isosceles triangle be obtuse. Likewise, the pictured scalene triangle is also obtuse, but that does not mean all scalene triangles are obtuse.

**Even though a triangle may fall into more than one of the categories,
but be sure treat each category according to its own properties.**

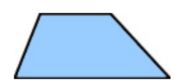
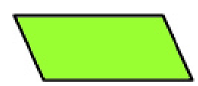
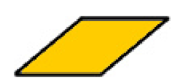
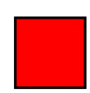
Another subtle, but important concept appears in the picture of the right triangle. Notice that there is a small square appearing in the right angle of the triangle. Since all the angles in a square measure 90°, that small square is there to indicate that angle measures 90°.

In terms of angles, regardless of the size and type of triangle, the sum of all three interior angles (angles *inside* the polygon) must be 180°. In an equilateral triangle, since all three angles are equal, each one measures 60°. For an obtuse triangle, since one angle is greater than 90°, the other two must both be less than 90°. Additionally, if a triangle has two or more angles that are equal in measure, then the sides opposite from those angles are also equal in measure. We see this with isosceles and equilateral triangles.

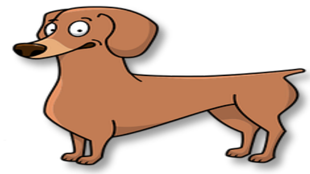
Last, but not least, we have the **Triangle Inequality**. It states, in order to construct a triangle from 3 line segments, the sum of the measures of any two of the segments must be greater than the measure of the third segment. For example, we cannot construct a triangle with segments of lengths 3 cm, 4 cm, and 9 cm, because the sum of 3 and 4 is NOT greater than 9. If you don't believe it, grab a ruler and try it!

Types of Quadrilaterals and Their Properties

Quad means "four" and lateral means "side." Thus, a **quadrilateral** is a four-sided polygon.

Quadrilateral	Properties	Picture
Trapezoid	*Exactly* one pair of opposite sides is parallel.	
Parallelogram	Both pairs of opposite sides are parallel.	
Rectangle	All angles measure 90°.	
Rhombus	All sides are equal in length.	
Square	All sides are equal in length, and all angles measure 90°.	

Like triangles, many quadrilaterals may fall in to more than one category. When we name a given quadrilateral, we should be as specific as possible. If we see a dachshund walking down the street, we would not be wrong to call it a dog. However, if we know the dog is a dachshund, it is better to use the more specific classification.



For trapezoids, some people use the definition of "*at least* one pair of parallel sides." With that definition, parallelograms, rectangles, rhombuses and squares would all be considered trapezoids. For our purposes, we will stick with the exclusive definition and say a trapezoid has *exactly* one pair of parallel sides. Thus, for our definition, rectangles, rectangles, parallelograms, rhombuses and squares are not trapezoids.

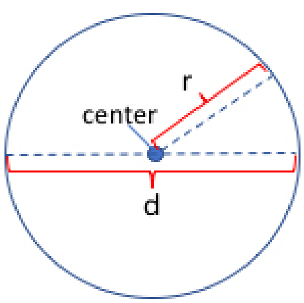
All rectangles are parallelograms. Squares satisfy the definitions for rhombuses, rectangles, and parallelograms! That is, even though all rectangles are parallelograms, if the angles all measure 90°, we should call it a rectangle, unless, of course, it is a square. ☺

Like triangles, quadrilaterals may fall into more than one category.

In every quadrilateral, the sum of the interior angles is always 360°. In squares and rectangles, since the angles are all 90°, they are all equal. In parallelograms – which includes and rectangles, rhombuses and squares – opposite interior angles are equal in measure.

Circles

All of us can identify a **circle** by sight, so we will focus on a few certain properties of circles.



The formal definition of a circle is all the points in a plane that are the same distance away from a single point, which is the **center** of the circle. Furthermore, that distance from the center to a point on the circle is called the **radius** of the circle, and is indicated with the letter r.

The **diameter**, d, of a circle, which is twice the radius, is the distance across a circle as measured through the circle's center, and is indicated with the letter d. The distance around a circle is called its **circumference**, C.

Example 1: Find the radius of a circle with diameter 12 cm.

The radius is half the diameter, so r = 6 cm.

Example 2: Find the diameter of a circle with a radius of 9 ft.

The diameter is twice the radius, so d = 18 ft.

Use a piece of string and a metric ruler to measure the circumference and diameter of three small, circular objects (e.g. a bottle cap). Record your measures in the following table. Then, in each case, compute C/d (rounded to the nearest hundredth).

Object	Circumference, C	Diameter, d	C/d
____________________________	______ cm	______ cm	________
____________________________	______ cm	______ cm	________
____________________________	______ cm	______ cm	________

For each of the circular objects, you should have found C/d to be close to 3.14. This value is represented by the lower-case Greek letter **pi,** π. If you did everything accurately in the above experiment, you should have found the value of C/d to be really close to pi for each one.

Indiana House Bill #246

In 1897, **Indiana House Bill No. 246** attempted to set the value of π to an incorrect rational approximation. The bill, written by amateur mathematician and medical doctor, **Edwin Goodwin**, was so poorly crafted it even contradicted itself, implying three different values for π. Nevertheless, Dr. Goodwin succeeded in getting his State Representative, Taylor Record, to introduce the bill, under the agreement that the State of Indiana could use the information free of charge, but the rest of the country would have to pay him royalties for its usage.

The bill, which, apparently, no Representative understood, passed through the Indiana House of Representatives unanimously (67-0), and was sent to the Senate for approval. Fortunately, during the House's debate on the Bill, **Purdue University** Mathematics Professor **Clarence Waldo** was present and became horrified. After the debate, a Representative offered to introduce Professor Waldo to Dr. Goodwin. Prof. Waldo declined by stating that he was already acquainted with as many crazy people as he cared to know. Later that evening, Professor Waldo informed the members of the Indiana Senate of the "merits" of the bill. The next day, after some good-natured ridicule at the expense of their colleagues in the House, the Senate moved the bill to an obscure committee and let it die a painless death.

Solids

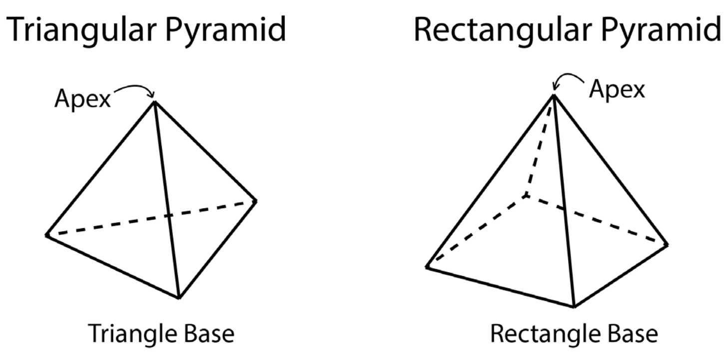
The world around us has three basic dimensions: height, width and depth. From shoe boxes, prisms, to soup cans, to ice cream cones, we will look at how we recognize, classify, and perform computations involving some of these shapes.

Boxes

While it is possible to have irregularly-shaped boxes, we will stick to **boxes** with 6 sides that are all rectangles. Each side is called a **face** and, for our purpose, we will work with boxes in which the opposite faces are equal in shape and size. A box has 8 corners, which are called **vertices**, and 12 **edges**. If the faces are all squares, we can further classify the box as a **cube**.

Pyramids

Like boxes, **pyramids** can come in many different shapes and sizes. Pyramids have a polygonal **base** with **triangular faces** that connect at a single point, called the **apex**. We will deal with **right pyramids**, which have the apex directly above the center of the base. We can further classify pyramids according to the shape of the polygon used for the base. We will confine our discussions to **triangular** and **rectangular pyramids**.

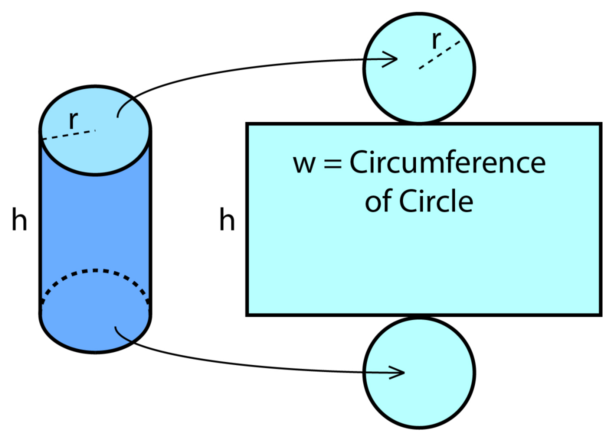


Like boxes, pyramids have **vertices** (the apex is a vertex) and **edges**, but the number of vertices and edges depends on the type of pyramid. Also, when drawing a pyramid or any three-dimensional figure, if we chose to indicate all of the edges, it is customary to make the hidden edges with dashed lines.

Cylinders

Tubes, pipes, and cans are all types of **cylinders**. Similar to our discussions about boxes and pyramids, we will deal with **right circular cylinders**.

Since it is round, it can be thought of as having circles for the top and bottom of the figure. If the side was cut from the top to the bottom, unrolled and laid flat, it would appear as a rectangle. For that "rectangle" the height is the same as the height of the cylinder, while the width is equal to the circumference of the circles.

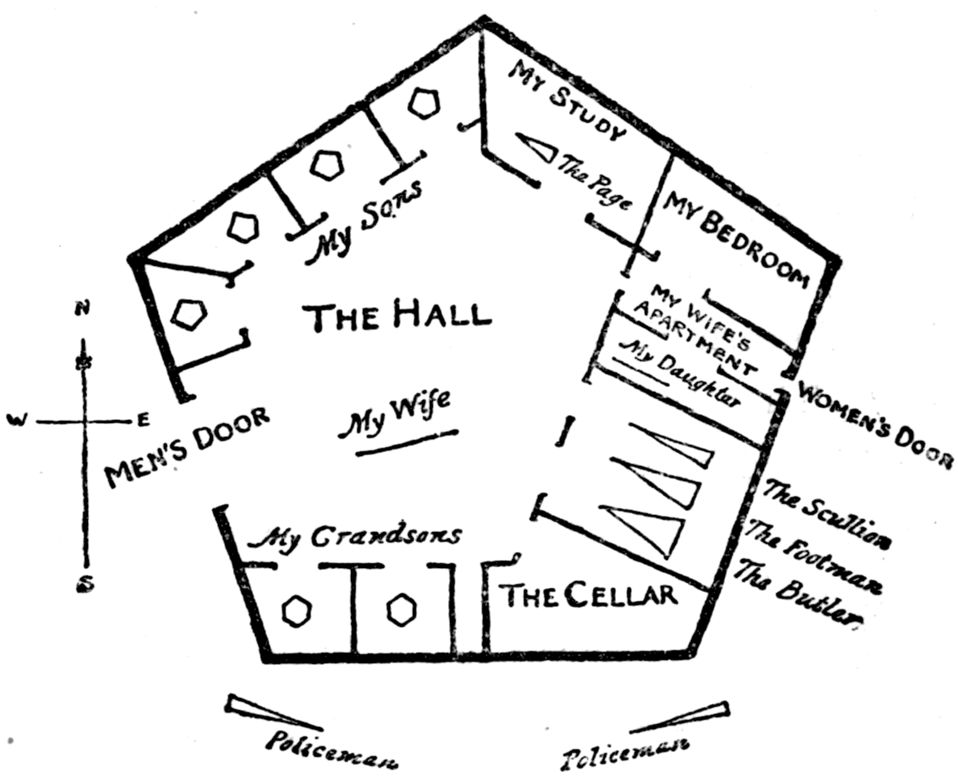


Polygons on the Internet

We opened this section with a partial list of common (and not-so-common) polygons. There are many, many more polygons than the handful we listed. If you are curious about septagons (7 sides), chiliagons (1000 sides), or megagons (1,000,000 sides), you can read all about them on Wikipedia at https://en.wikipedia.org/wiki/List_of_polygons.

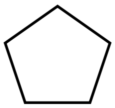
What would life be like in a two-dimensional world? In 1884 (yes, 1884), Englishman Edwin Abbot wrote a satirical novel called *Flatland: A Romance of Many Dimensions*. The book describes life, society, and the world of a square named, appropriately enough, "A Square." You can read the book for free online by going to Google Play and searching the Books for "Flatland."

If you do read *Flatland*, be forewarned that many readers find the description of the square's world to be very sexist. In it, the male figures are polygons where the greater number of sides defines a higher social class, while the women are straight lines. In reprintings of the book, Abbot even addressed this by pointing out the aim was not sexist; it was merely the status of the world being described. Remember, it was written in 1884.

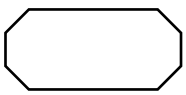


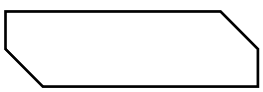
Section 2.2 Exercises

For Exercises #1-9, name the polygon.

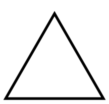
1. 

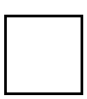
2. 

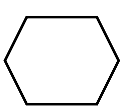
3. 

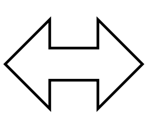
4. 

5. 

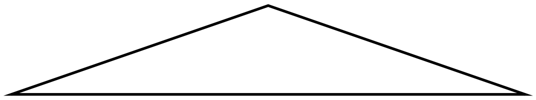
6. 

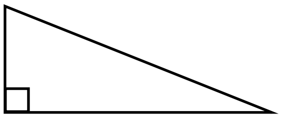
7. 

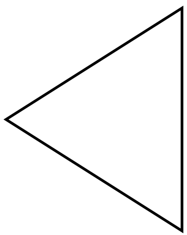
8. 

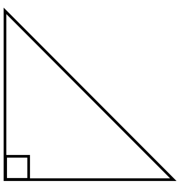
9. 

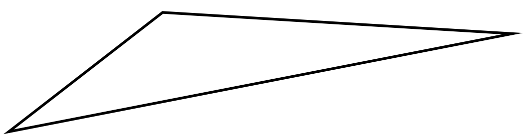
For Exercises #10-16, classify the triangles as equilateral, isosceles, scalene, right, acute and/or obtuse. Some triangles may have more than one classification. Be sure to indicate all that apply.

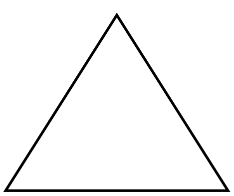
10. 

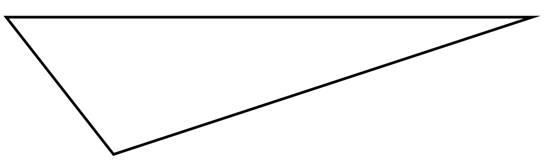
11. 

12. 

13. 

14. 

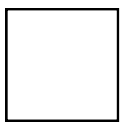
15. 

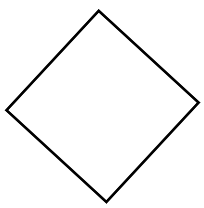
16. 

For Exercises #17-23, classify the quadrilaterals as a trapezoid, parallelogram, rectangle, rhombus and/or square. Some figures may have more than one classification. Be sure to indicate all that apply.

17. 

18. 

19. 

20. 

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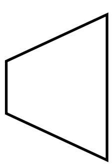
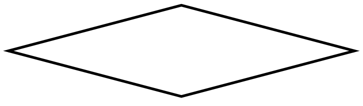
22. 

23. 

24. What is the radius of a circle with a diameter of 5 mm?

25. What is the radius of a circle with a diameter of 18 cm?

26. What is the diameter of a circle with a radius of 15 in?

27. What is the diameter of a circle with a radius of 7 ft?

28. How many edges are on a cube?

29. How many vertices are on a triangular pyramid?

30. How many edges are on a rectangular pyramid?

31. How many triangles does it take to make a triangular pyramid?

32. How many rectangles are needed to make a rectangular pyramid?

Section 2.2 Exercise Solutions

1. Pentagon
2. Triangle
3. Octagon
4. Hexagon
5. Quadrilateral
6. Triangle
7. Quadrilateral
8. Hexagon
9. Decagon
10. Isosceles, Obtuse

11. Scalene, Right
12. Isosceles, Equilateral, Acute
13. Isosceles, Right
14. Scalene, Obtuse
15. Isosceles, Equilateral, Acute
16. Scalene, Obtuse
17. Parallelogram, Rectangle
18. Trapezoid
19. Parallelogram, Rectangle, Rhombus, Square
20. Parallelogram, Rectangle, Rhombus, Square
21. Parallelogram
22. Trapezoid
23. Parallelogram, Rhombus
24. r = (1/2)d = (1/2)(5 mm) = 2.5 mm
25. r = (1/2)d = (1/2)(18 cm) = 9 cm

26. d = 2r = 2(15 in) = 30 in

27. d = 2r = 2(7 ft) = 14 ft

28. 12

29. 4

30. 8

31. 4

32. 1 (The faces are triangles.)

2.3: PERIMETER, AREA & VOLUME

The **perimeter** of any flat geometric figure is defined as the distance around the figure. Although we may be presented with different formulas that get used for different figures, the perimeter of a given figure will still always be the distance around it.

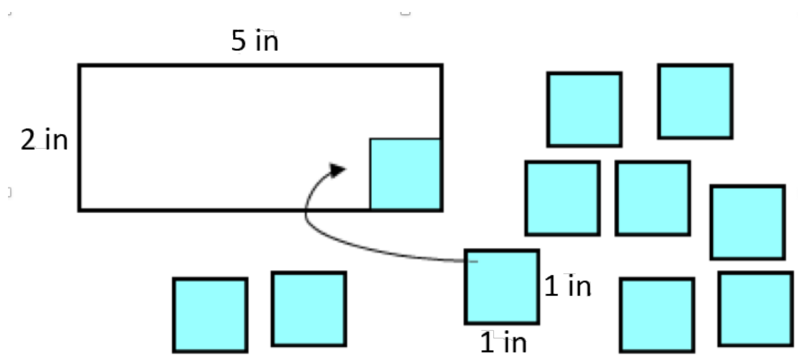
The perimeter of any figure is the linear distance around the figure.

The **area** of a flat geometric figure is defined as the amount of surface the figure covers.

The area of a figure is the amount of surface it covers.

Perimeter and Area of Rectangles

For rectangle and squares, area can be viewed as the number of square units that can be enclosed by the figure. In fact, for that very reason, area is always stated in square units. For example, imagine a square that measures 1 inch on each side. How many of those 1-in squares – also called “square inches” - can fit inside a rectangle that measures 2 inches wide by 5 inches long? If you're not sure, draw a picture.



When we are computing areas and perimeters we need to pay attention to the units of measure. If no units are stated, then only numeric answers should be given.

If units are stated, we must include the appropriate unit with our answers.
Linear units should accompany the linear measures of perimeter.
Areas should be stated in square units.

Example 1: A rectangular room measures 9 feet by 12 feet. How many square feet of carpet are needed to cover the floor? Also, how many feet of baseboard trim will be needed to go around the base of the room? Assume the door to the room is 3 ft wide.

For the carpet, we need the area of the floor: 9 ft × 12 ft = 108 ft^2
We need 108 square feet of carpet.

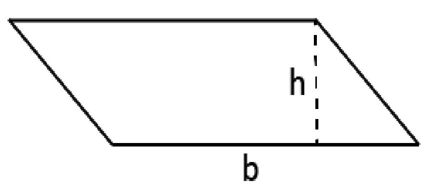
For the baseboard trim, we need to find the perimeter of the room, less the width of the door:
9 ft + 12 ft + 9 ft + 12 ft - 3 ft (for the doorway) = 39 ft
We need 39 feet of baseboard trim.

Notice how we were able to perform the previous example without the use of formulas. In the case of a perimeter, whether we have a rectangle or not, all we need to do is add up the lengths of all the sides of the figure. When dealing with areas, we have already seen the area of a rectangle is the product of the length and width. In formula form, if we use L for the length and W for the width, the formula is $A = L \times W$. Keep in mind, if we understand the origin of the formula, there is no need to memorize it.

Area of a Rectangle = Length × Width

Perimeter and Area of Parallelograms

A parallelogram is a four sided-figure in which both pairs of opposite sides are equal in length and parallel, but the angles are not necessarily right angles. Just like any flat geometric figure, the perimeter of a parallelogram is the distance around it. To find the area, we first need to identify the lengths of the base and height of the parallelogram.

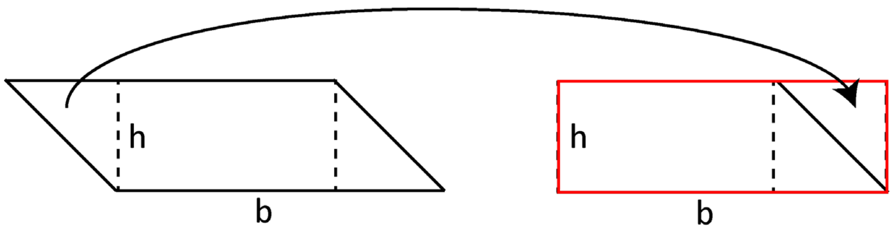


The **base**, b, of the parallelogram is the length of the bottom side. Actually, we can use any side, but, for simplicity, let's stick with the bottom. The **height**, h, of a parallelogram is the perpendicular distance from the base to the opposite side. This is usually represented by a dashed line drawn perpendicular to the base.

COMMON MISTAKE

Do not confuse the height, h, with the length of a side.
That is why we use a dashed line instead of a solid line.

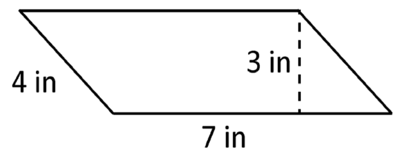
To determine the area of a parallelogram, imagine cutting off the triangular region on one side and moving it to the other side.



The resultant figure would be a rectangle of dimensions b and h. Thus, the area of the parallelogram would be the product of the base and the height, or A = bh.

Area of a Parallelogram = Base × Height

Example 2: Find the perimeter and area of the parallelogram.



The perimeter, P, is the sum of the four side lengths.

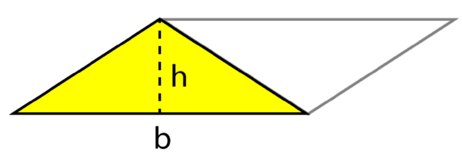
P = 4 in + 7 in + 4 in + 7 in = 22 in (Do not include the 3 in, as it is not a side measure!)

The area is the product of the base and the height. In this case the base is 7 inches and the height is 3 inches.

A = (7 in)(3 in) = 21 in^2

Perimeter and Area of Triangles

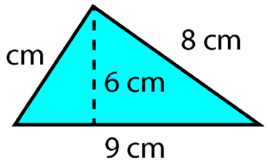
When examining triangles, the perimeter is just the sum of the three side lengths. For the area, we need to recognize a triangle is just half of a parallelogram when we cut it along the diagonal.



The height, h, and base, b, lengths are the same as with a parallelogram, but since we simply have half a parallelogram, the area of a triangle is half that of the parallelogram with the same base and height. That is, A = (1/2)(b)(h).

Area of a Triangle = (1/2)(Base × Height)

Example 3: Find the perimeter and area of the triangle.



The perimeter: P = 7 cm + 8 cm + 9 cm = 24 cm
(Do not include the 6 cm, as it is not a side measure!)

The area: A = (1/2)(9 cm)(6 cm) = 27 cm^2

Circumference and Area of Circles

Using r for the radius, and d for the diameter, the formulas for the area and circumference of a circle are:

Formula for the Area of a Circle: $A = \pi r^2$.

Formula for the Circumference of a Circle: $C = \pi d$.
Or, since d = 2r, we often see this as $C = 2\pi r$.

Also, to approximate the calculations, it is customary to use $\pi = 3.14$.

Example 4: Find the area and circumference for a circle of radius 3 inches. Be sure to label your answers. Use $\pi = 3.14$, and round your answers to the nearest hundredth.

Area, $A = \pi r^2 = (3.14)(3\text{ in})^2 = (3.14)(9\text{ in}^2) = 28.26\text{ in}^2$

Circumference, $C = 2\pi r = 2(3.14)(3\text{ in}) = 18.84\text{ in}$

Example 5: Find the area and circumference of the circle. Be sure to label your answers. Use $\pi = 3.14$, and round your answers to the nearest tenth.

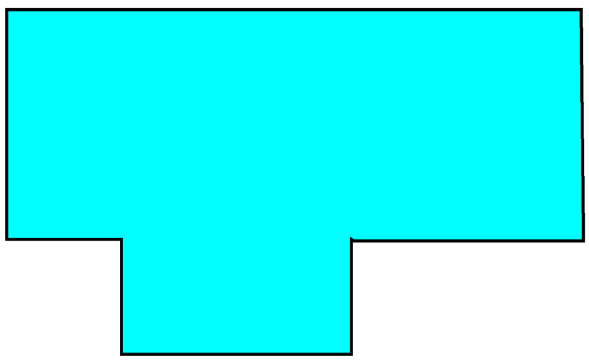
Area, $A = \pi r^2 = (3.14)(15\text{ cm})^2 = (3.14)(225\text{ cm}^2) = 706.5\text{ cm}^2$

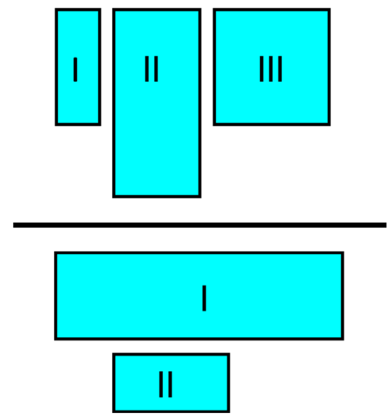
Circumference, $C = 2\pi r = 2(3.14)(15\text{ cm}) = 94.2\text{ cm}$

Perimeter and Area of Composite Figures

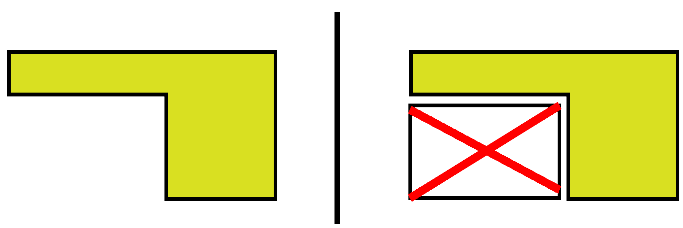
A **composite figure** is made up of two or more smaller figures. Like always, the perimeter of such a figure is just the distance around the figure.

For the area, if the composite contains only right angles, we can picture cutting the figure into squares and rectangles. Then, we can find the areas of the smaller pieces and add them together to get the total area, which we will write as A_{Total}. Do remember, though, composite figures can also be made of other shapes, as well.





Alternatively, for the area, sometimes, we can envision a composite figure as a larger figure with a section cut away. In such a case, the area of the composite figure would be the area of the larger figure less the area of the cutout.

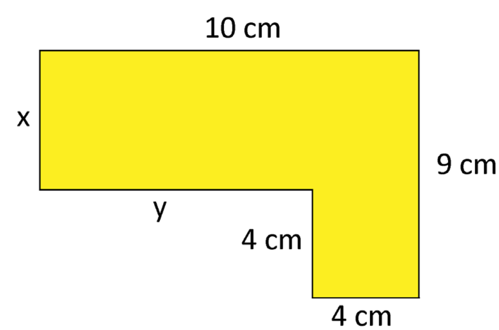


It is important to note that, for the perimeter, we sum up *only* the sides that form the border of the given composite figure. We should not try to break apart a figure to find the perimeter, but we may need to use some critical thinking skills to determine the lengths of unidentified sides.

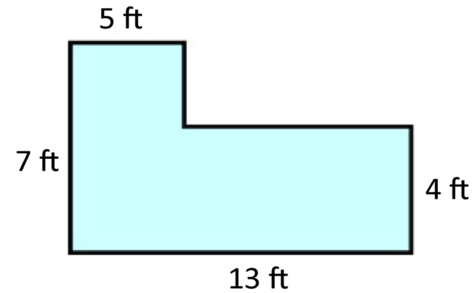
Example 6: Find the side lengths x and y.

For x, we see the entire height of the figure is 9 cm, and the cutout piece is 4 cm high.
That means 9 cm = 4 cm + x. Which tells us x = 5 cm.

For y, we see the entire width is 10 cm, the cutout is y, and the remaining piece width is 4 cm.
That means 10 cm = y + 4 cm. That tells us y = 6 cm



Example 7: Find the area and perimeter of the figure.



Before we can accurately compute the area and perimeter, we must first determine the lengths of the two unlabeled sides.

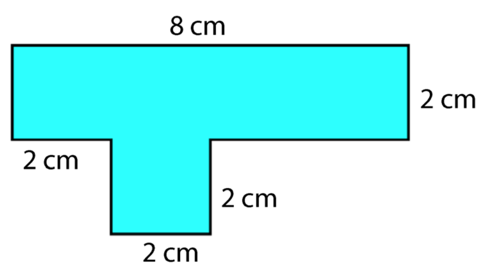
For the missing horizontal measure, we need to notice the length of the bottom is 13 ft, and the length of the top-most side is 5 ft. That means the missing horizontal measure must be 8 ft.

Likewise, the missing vertical measure can be found to be 3 ft.

Next, for the area, we can imagine the composite as 7 ft × 13 ft rectangle with a 3 ft × 8 ft cut out of it. Thus, the total area, $A = (7\text{ ft})(13\text{ ft}) - (3\text{ ft})(8\text{ ft}) = 91\text{ ft}^2 - 24\text{ ft}^2 = 67\text{ ft}^2$.

For the perimeter, we add together the lengths of all six sides. Be careful not to forget the lengths of the unlabeled sides. Starting at the top and going clockwise, we find the perimeter, P = 5 ft + 3 ft + 8 ft + 4 ft + 13 ft + 7 ft = 40 ft.

Example 8: Find the area of the composite figure.



The figure can be viewed as a 2 cm × 8 cm rectangle on top of a 2 cm × 2 cm square.

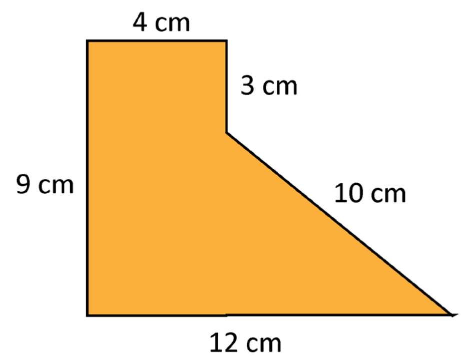
$A_{Total} = A_{Rectangle} + A_{Square} = 16\text{ cm}^2 + 4\text{ cm}^2 = 20\text{ cm}^2$

Alternatively, we could view it as a full 8 cm × 4 cm rectangle with a 2 cm × 2 cm cutout in the lower left, and a 2 cm × 4 cm cutout in the lower right. In that case:

$A_{Total} = 32\text{ cm}^2 - 4\text{ cm}^2 - 8\text{ cm}^2 = 20\text{ cm}^2$

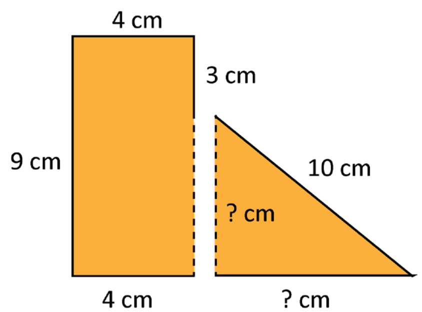
So far, the composite figures were just mixture of squares and rectangles. We can also have triangular shapes, and even circular parts thrown into the mix. We still imagine the composite cut into smaller pieces, and we may also need to deduce some lengths to figure out the total area and perimeter

Example 9: Find the area and perimeter of the figure.



For the area, we should imagine the figure as a 4 cm by 9 cm rectangle with a triangular piece next to it.

BE CAREFUL! The base length of that triangle part is not 12 cm; it is only 8 cm. Also, how do we find the height of the triangular piece?



Since the length of the base for the whole composite figure is 12 cm, and 4 cm of that is composed of the rectangle, the triangle-shaped region has a base of 8 cm. For the height of the triangular region, we see the entire height of the composite figure is 9 cm, and the triangle height is 3 cm less than that. So, the height of the triangle is 6 cm.

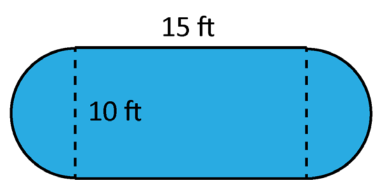
Now, taking the entire composite figure in mind,

$A_{Total} = A_{Rectangle} + A_{Triangle} = (4\text{ cm})(9\text{ cm}) + (1/2)(8\text{ cm})(6\text{ cm}) = 36\text{ cm}^2 + 24\text{ cm}^2 = 60\text{ cm}^2$.

For the perimeter, start at the top and go clockwise around the figure. Be sure to remember the two lengths we had to deduce are NOT part of the perimeter.

P = 4 cm + 3 cm + 10 cm + 12 cm + 9 cm = 38 cm.

Example 10: Find the area for the figure. Use $\pi = 3.14$, and round your answers to the nearest tenth, if necessary.



This shape can be viewed as a 10 ft $\times$ 15 ft rectangle, with a semicircle at each end.

If the two semicircles are detached and placed together, they would form a whole circle with a 10-ft diameter. That makes the area of the composite equal to the sum of the circle's area and the rectangle's area. Be sure to use the radius, not the diameter in the computation.

$A_{Total} = A_{Circle} + A_{Rectangle} = (3.14)(5\text{ ft})^2 + (10\text{ ft})(15\text{ ft}) = 78.5\text{ ft}^2 + 150\text{ ft}^2 = 228.5\text{ ft}^2$

Volume and Surface Area of a Box

Linear distances - such as feet (ft), inches (in), and miles (mi) – are used quite frequently in our daily lives. Likewise, areas – such as square feet (ft^2), square inches (in^2) and square miles (mi^2) – are pretty common, too. Additionally, there is another relatively common measure: **Volume**. Refrigerators have capacity stated in cubic feet (ft^3), we pour concrete in cubic yards (yd^3), and we buy gallons of milk, water and gasoline. Volumes point to the third dimension in our three-dimensional world.

Surface area refers to the total area of all of the faces of the object.

Similar to how the area of a rectangle was found as the length times the width, the volume a box is found by multiplying the length by the width by the height. This can also be thought of as the area of the base, times the height.

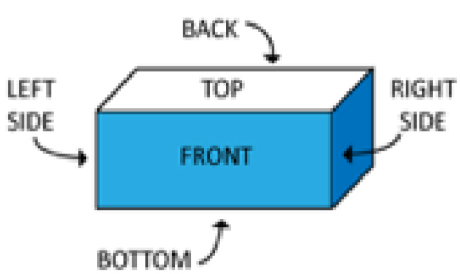
Volume of a Box = Length × Width × Height

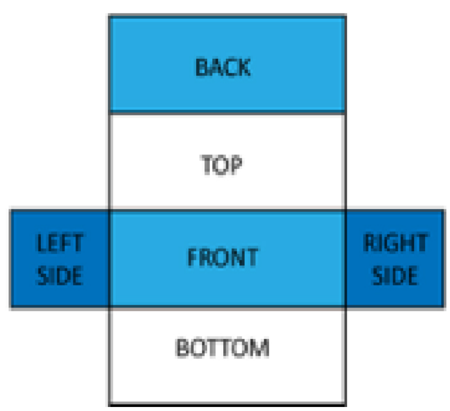
Be sure to pay attention to the unit of measure. Many volumes are stated in imperial measures like gallons and pints, or in metric measures like liters and milliliters. If, however, a volume is computed from the product of linear dimensions, it needs to have a cubic unit. That is, if we compute the volume by finding the product of a length, width and height that were all measured in centimeters, the final unit of measure on the volume would be cm^3. This is just like multiplying $a \times a \times a$ to get a^3, $cm \times cm \times cm$ is cm^3,

Example 10: Find the volume of a box that has a length of 5 inches, a width of 8 inches, and a height of 4 inches.

$V_{Box} = L \times W \times H = 5 \text{ in} \times 8 \text{ in} \times 4 \text{ in} = 160 \text{ in}^3$

If the box is sliced open along few of its edges, unfolded it, and laid flat, we would see 6 rectangles. If we find the area of each of those 6 rectangles and add them together, we have the surface area of the whole box.





Surface Area = Sum of the Areas of the Faces

Example 11: Mason wants to paint the outside of a wooden crate. If the crate is 4 feet long by 5 feet wide by 2 feet deep, how much paint is needed?

The front of the crate is 5 feet wide × 2 feet high.
A_{Front} = (5 ft)(2 ft) = 10 ft^2.
The back of the box is the same as the front.

The top of the box is 5 feet wide × 4 feet long. A_{Top} = (5 ft)(4 ft) = 20 ft^2.
The bottom of the box is the same as the top.

The right side of the box is 4 feet long × 2 feet high. A_{Right} = (4 ft)(2 ft) = 8 ft^2.
The left side of the box is the same as the right side.

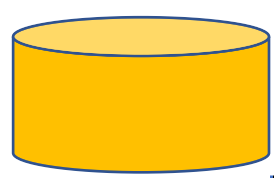
$$\begin{aligned}\text{Surface Area} &= A_{Front} + A_{Back} + A_{Top} + A_{Bottom} + A_{Right} + A_{Left} \\ &= 10\text{ ft}^2 + 10\text{ ft}^2 + 20\text{ ft}^2 + 20\text{ ft}^2 + 8\text{ ft}^2 + 8\text{ ft}^2 \\ &= 76\text{ ft}^2\end{aligned}$$

Mason needs 76 square feet of paint.

Volume and Surface Area of a Cylinder

The volume of a cylinder is similar to the volume of a box: We find the area of the base, and then multiply by the height. In this case, the area of the base is just the area of the circle that is the bottom of the cylinder.

Volume of a Cylinder = Area of the Base × Height

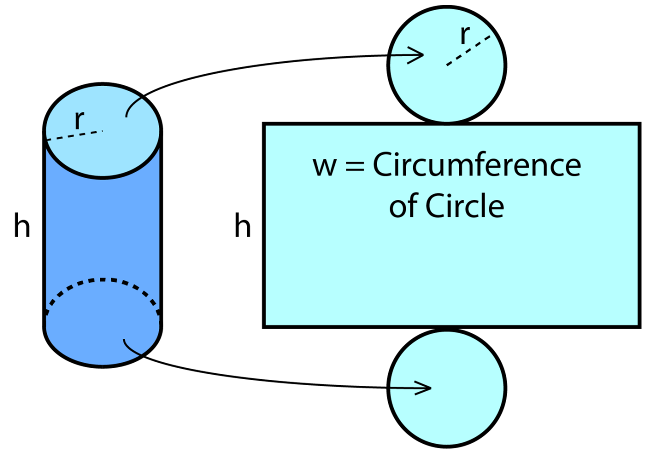


Example 12: Find the volume of a cylinder with a radius of 30 mm, and a height of 22 mm. Use π = 3.14 and round your answer to the nearest tenth.

First, find the area of the base, and then multiply by the height.

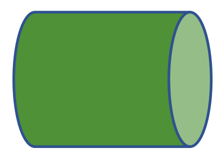
Volume = A_{Base} × Height = $\pi r^2 \times h$ = (3.14)(30 mm)^2 × 22 mm = 62,172 mm^3

For the surface area of a cylinder, like we did for a box, we need to open it up and lay it flat. Once that is done, we see the cylinder is made up of two circles and a rectangle. Additionally, the rectangle has a width that is equal to the circumference of the circular base.

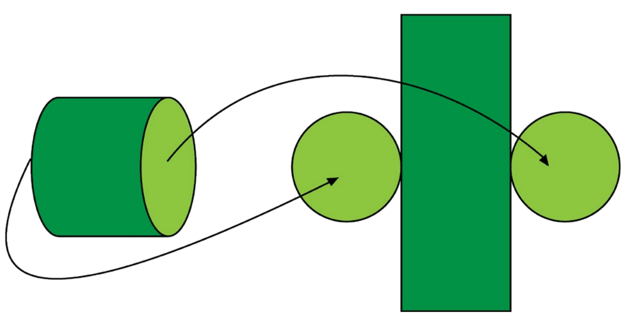


Surface Area of a Cylinder = 2 × Area of the Base + Area of Rectangular Side

Example 13: Find the surface of a cylinder with a radius of 12 cm, and a height of 15 cm. Use π = 3.14 and round your answer to the nearest hundredth.



Even though the pictured cylinder is sideways, the base is still the circular part, which means the "height" is the distance between the two circular ends. Additionally, when we find the area of one circle, we just multiply by 2 to account for the area of the other circle.



Surface Area $= 2A_{\text{Circle}} + A_{\text{Rectangle}}$

$= 2\pi r^2 + 2\pi rh$

$= 2(3.14)(12\text{ cm})^2 + 2(3.14)(12\text{ cm})(15\text{ cm}) = 2034.72\text{ cm}^2$

Using Your Calculator

Many scientific calculators have a π key or function. Throughout this section we have used the approximation $\pi = 3.14$. If we use the calculator key/function, we would be using a much more precise approximation – typically $\pi = 3.141592654$, or better. Do realize, however, the solutions presented for the exercises have used $\pi = 3.14$. So, if you see the more precise value, your answers may be a little off.

On the Internet

There are many online calculators and phone apps tailored for specific tasks, and computing areas and volumes are no exception. Do realize, however, many of those calculators were developed by students completing assignments, which means they may or may not function correctly. If you use one of those tools, it may be worth double-checking the computations until you develop a bit of confidence.

Section 2.3 Exercises

1. Find the perimeter and area of a square that has a side length 5 inches.

2. Find the perimeter and area of a square with a side length of 3 miles.

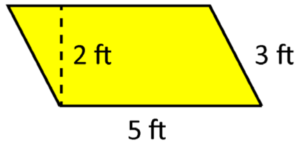
3. Find the perimeter and area of a rectangle with a length of 2 m and a width of 3 m.

4. Find the perimeter and area of a rectangle with a length of 2 ft and a width of 9 ft.

5. Find the perimeter and area of a rectangle with a length of 5 cm and a width of 12 cm.

6. Find the perimeter and area of a rectangle with a length of 6 ft and a width of 8 ft.

7. Find the perimeter and area of the parallelogram.

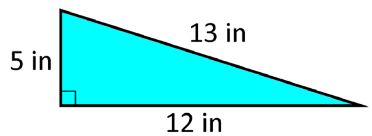


8. Find the perimeter and area of the parallelogram.

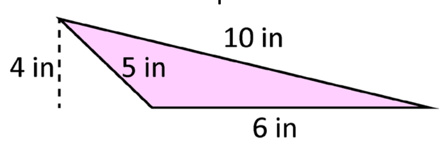
9. Find the area of a triangle with a base of 12 inches and a height of 3 inches.

10. Find the area of a triangle with a base of 7 meters and a height of 4 meters.

11. Find the area and perimeter of the triangle.



12. Find the area and perimeter of the triangle.

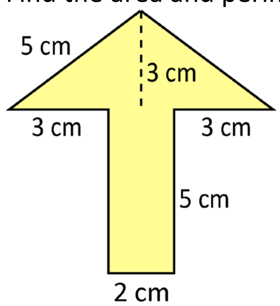


13. A rectangular room measures 10 ft by 12 ft. How many square ft of tile are needed to cover the floor?

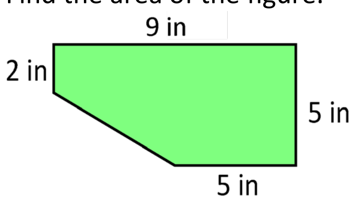
14. John's kitchen is rectangular and measures 11 feet wide by 13 feet long, and he has a 3 feet by 4 feet island in the middle of the kitchen. Assuming no tiles are placed under the island, how many square feet of tile will John need to tile the floor?

15. A skirt is to be placed around a rectangular table that is 6 feet long and 2 feet wide. How long does the table skirt need to be, if it is to be wrapped around the table?

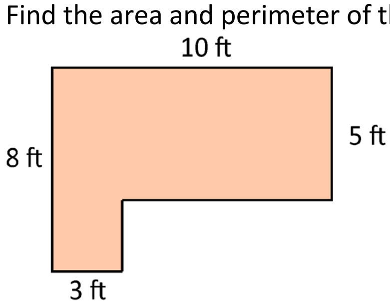
16. Find the area and perimeter of the figure.



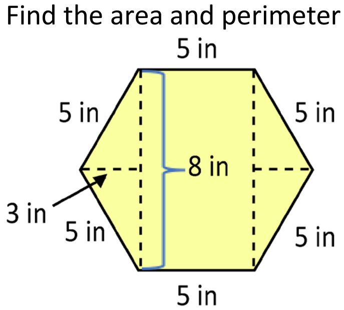
17. Find the area of the figure.



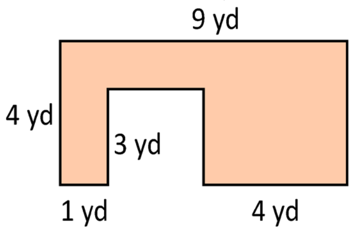
18. Find the area and perimeter of the figure.



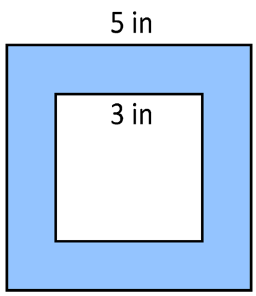
19. Find the area and perimeter of the figure.



20. Find the area and perimeter of the figure.



21. Find the area of the shaded region in the figure.



22. What is the radius of a circle with diameter 12 in?

23. What is the radius of a circle with a diameter of 25 ft?

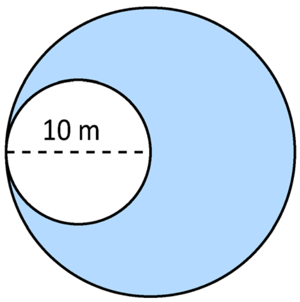
24. Find the area and circumference of a circle with radius 10 cm. Use π = 3.14, and round your answers to the nearest hundredth.

25. Find the area and circumference of a circle with radius 20 cm. Use π = 3.14, and round your answers to the nearest hundredth.

26. Find the area and circumference of a circle with radius 12 m. Use $\pi = 3.14$.

27. Find the area and circumference of a circle with radius 15 mm. Use $\pi = 3.14$.

28. Find the area of the shaded region in the following figure. Use $\pi = 3.14$, and round your answers to the nearest hundredth.



29. A table skirt is to be wrapped around a circular table with a diameter of 6 feet. How long does the skirt need to be, if it is to be wrapped around the table? Use $\pi = 3.14$, and round your answers to the nearest whole foot.

30. Find the volume and surface area of a box that has a length of 3 cm, a width of 9 cm, and a height of 5 cm.

31. Find the volume and surface area of a box that has a length of 7 in, a width of 2 in, and a height of 6 in.

32. Find the volume and surface area of a cylinder that has a base radius if 6 m and a height of 5 m. Use π = 3.14, and round your answers to the nearest hundredth.

33. Find the volume and surface area of a cylinder that has a base radius if 7 in and a height of 6 in. Use π = 3.14, and round your answers to the nearest hundredth.

Section 2.3 Exercise Solutions

1. $P = (4)(5 \text{ in}) = 20 \text{ in}$, $A = (5 \text{ in})^2 = 25 \text{ in}^2$

2. $P = (4)(3 \text{ mi}) = 12 \text{ mi}$, $A = (3 \text{ mi})^2 = 9 \text{ mi}^2$

3. $P = 2 \text{ m} + 3 \text{ m} + 2 \text{ m} + 3 \text{ m} = 10 \text{ m}$, $A = (2 \text{ m})(3 \text{ m}) = 6 \text{ m}^2$

4. $P = 2 \text{ ft} + 9 \text{ ft} + 2 \text{ ft} + 9 \text{ ft} = 22 \text{ ft}$, $A = (2 \text{ ft})(9 \text{ ft}) = 18 \text{ ft}^2$

5. $P = 5 \text{ cm} + 12 \text{ cm} + 5 \text{ cm} + 12 \text{ cm} = 34 \text{ cm}$, $A = (5 \text{ cm})(12 \text{ cm}) = 60 \text{ cm}^2$

6. $P = 6 \text{ ft} + 8 \text{ ft} + 6 \text{ ft} + 8 \text{ ft} = 28 \text{ ft}$, $A = (6 \text{ ft})(8 \text{ ft}) = 48 \text{ ft}^2$

7. $P = 3 \text{ ft} + 5 \text{ ft} + 3 \text{ ft} + 5 \text{ ft} = 16 \text{ ft}$, $A = (5 \text{ ft})(2 \text{ ft}) = 10 \text{ ft}^2$

8. $P = 5 \text{ in} + 20 \text{ in} + 5 \text{ in} + 20 \text{ in} = 50 \text{ in}$, $A = (20 \text{ in})(4 \text{ in}) = 80 \text{ in}^2$

9. $A = (1/2)(12 \text{ in})(3 \text{ in}) = 18 \text{ in}^2$

10. $A = (1/2)(7 \text{ m})(4 \text{ m}) = 14 \text{ m}^2$

11. $A = (1/2)(12 \text{ in})(5 \text{ in}) = 30 \text{ in}^2$, $P = 5 \text{ in} + 13 \text{ in} + 12 \text{ in} = 30 \text{ in}$

12. $A = (1/2)(6 \text{ in})(4 \text{ in}) = 12 \text{ in}^2$, $P = 10 \text{ in} + 6 \text{ in} + 5 \text{ in} = 21 \text{ in}$

13. $A = (10 \text{ ft})(12 \text{ ft}) = 120 \text{ ft}^2$. 120 square feet of tile are needed.

14. $A_{\text{Total}} = A_{\text{Room}} - A_{\text{Island}} = 143 \text{ ft}^2 - 12 \text{ ft}^2 = 131 \text{ ft}^2$. 131 sq ft of tile are needed.

15. The skirt wraps around the perimeter of the table, which is 16 ft.

16. For the area, split the figure into a triangle of base 8 cm and height 3 cm, on top of a 2 cm by 5 cm rectangle. $A = (1/2)(8 \text{ cm})(3 \text{ cm}) + (2 \text{ cm})(5 \text{ cm}) = 22 \text{ cm}^2$. For the perimeter, make sure you identify the two unlabeled sides, which are both 5 cm. Add the side lengths starting at the top and moving clockwise. $P = 5 \text{ cm} + 3 \text{ cm} + 5 \text{ cm} + 2 \text{ cm} + 5 \text{ cm} + 3 \text{ cm} + 5 \text{ cm} = 28 \text{ cm}$.

17. Picture the figure as a 9 in by 5 in rectangle with a triangle cut off the lower left corner. The base of the triangle is 4 in and the height is 3 in. $A_{\text{Total}} = (9 \text{ in})(5 \text{ in}) - (1/2)(4 \text{ in})(3 \text{ in}) = 39 \text{ in}^2$.

18. For the area, picture the figure as a 10 ft by 8 ft rectangle with a smaller 7 ft by 3 ft rectangle cut out of the lower right corner. $A_{\text{Total}} = (10 \text{ ft})(8 \text{ ft}) - (7 \text{ ft})(3 \text{ ft}) = 59 \text{ in}^2$. For the perimeter, starting in the upper left corner and moving clockwise, $P = 10 \text{ ft} + 5 \text{ ft} + 7 \text{ ft} + 3 \text{ ft} + 3 \text{ ft} + 8 \text{ ft} = 36 \text{ ft}$.

19. For the area, imagine the figure as a 5 in by 8 in rectangle sandwiched between two sideways triangles that have a base of 8 in and a height of 3 in. $A_{\text{Total}} = A_{\text{Rectangle}} + 2A_{\text{Triangle}}$. $A = (5 \text{ in})(8 \text{ in}) + 2(1/2)(8 \text{ in})(3 \text{ in}) = 40 \text{ in}^2 + 24 \text{ in}^2 = 64 \text{ in}^2$. For the perimeter, the exterior of the hexagon is six, 5 in side lengths. $P = 6(5 \text{ in}) = 30 \text{ in}$.

20. For the area, imagine the figure as a larger 9 yd by 4 yd with a 3 yd by 4 yd cutout. A = 36 yd^2 – 12 yd^2 = 24 yd^2. P = 9 yd + 4 yd + 4 yd + 3 yd + 4 yd + 3 yd + 1 yd + 4 yd = 32 yd.

21. This is a large 5 in square with a 3 in square cut out of the middle. A = $(5\text{ in})^2 - (3\text{ in})^2 = 16\text{ in}^2$.

22. The radius is half the diameter. r = 6 in.

23. The radius is half the diameter. r = 12.5 ft.

24. $A = \pi r^2 = (3.14)(10\text{ cm})^2 = 314\text{ cm}^2$. $C = 2\pi r = 2(3.14)(10\text{ cm}) = 62.8\text{ cm}$

25. $A = \pi r^2 = (3.14)(20\text{ cm})^2 = 1256\text{ cm}^2$. $C = 2\pi r = 2(3.14)(20\text{ cm}) = 125.6\text{ cm}$

26. $A = \pi r^2 = (3.14)(12\text{ m})^2 = 452.16\text{ m}^2$. $C = 2\pi r = 2(3.14)(12\text{ m}) = 75.36\text{ m}$

27. $A = \pi r^2 = (3.14)(15\text{ mm})^2 = 706.5\text{ mm}^2$. $C = 2\pi r = 2(3.14)(15\text{ mm}) = 94.2\text{ mm}$

28. This is a large circle with a radius of 10 m, with a small circle of radius 5 m cut out of it.
$A_{Total} = A_{Large} - A_{Small} = (3.14)(10\text{ m})^2 - (3.14)(5\text{ m})^2 = 314\text{ m}^2 - 78.5\text{ m}^2 = 235.5\text{ m}^2$

29. The skirt it equal to the circumference of the table, which is $C = \pi d = (3.14)(6\text{ ft}) = 18.84\text{ ft} \rightarrow 19\text{ ft}$.

30. $V = L \times W \times H = (3\text{ cm})(9\text{ cm})(5\text{ cm}) = 135\text{ cm}^3$. S. A. = $A_{Front} + A_{Back} + A_{Top} + A_{Bottom} + A_{Right} + A_{Left}$ = $27\text{ cm}^2 + 27\text{ cm}^2 + 45\text{ cm}^2 + 45\text{ cm}^2 + 15\text{ cm}^2 + 15\text{ cm}^2 = 174\text{ cm}^2$

31. $V = L \times W \times H = (2\text{ in})(7\text{ in})(6\text{ in}) = 84\text{ in}^3$. S. A. = $A_{Front} + A_{Back} + A_{Top} + A_{Bottom} + A_{Right} + A_{Left}$ = $14\text{ in}^2 + 14\text{ in}^2 + 42\text{ in}^2 + 42\text{ in}^2 + 12\text{ in}^2 + 12\text{ in}^2 = 136\text{ in}^2$

32. $V = (\pi r^2)h = (3.14)(6\text{ m})^2(5\text{ m}) = 565.2\text{ m}^3$. S.A. = $2\pi r^2 + (2\pi r)h = 2(3.14)(6\text{ m})^2 + (2)(3.14)(6\text{ m})(5\text{ m})$ = $226.08\text{ m}^2 + 188.4\text{ m}^2 = 414.48\text{ m}^2$.

33. $V = (\pi r^2)h = (3.14)(7\text{ in})^2(6\text{ in}) = 923.16\text{ in}^3$. S.A. = $2\pi r^2 + (2\pi r)h = 2(3.14)(7\text{ in})^2 + (2)(3.14)(7\text{ in})(6\text{ in})$ = $307.72\text{ in}^2 + 263.76\text{ in}^2 = 571.48\text{ in}^2$.

2.4: RATIOS, RATES & PROPORTIONS

Ratios vs. Rates

A **ratio** is a comparison between two quantities. A common size for a picture is 5×7. That ratio is 5 in wide by 7 in long. When put into a ratio, the units will cancel, just as if they were common factors. Ratios can be written as a fraction or a pair of numbers separated by a colon or the word "to."

Example 1: What is the ratio of 36 feet to 40 feet? Write the ratio as two numbers separated by a colon.

When writing the ratio, it is customary to reduce the numbers, as if they were in a fraction.

$$\frac{36\ ft}{40\ ft} = \frac{9 \cdot 4\ ft}{10 \cdot 4\ ft} = \frac{9}{10}$$

Notice that the unit of "ft" cancels, just as the factor of 4 does. Remember, a ratio compares two quantities. As per the instructions, this is 9:10.

Example 2: A school has an enrollment of 3500 students and a total of 140 faculty members. What is the school's student to teacher ratio, written as a fraction?

3500/140 reduces to 25/1.

Do, however, remember to write both quantities, even if the second number is a 1. Writing just "25" for a ratio is insufficient.

Ratios including units of measure, such as 35 mi/hr or $3.65/gallon, are called **rates**. Rates are simplified a little differently than ratios. First, the different units of measure will not cancel. Next, we typically want a "per unit" rate. A **unit rate** is a rate with a denominator of 1. And, since the unit rate can be obtained by simple division, the process is a little more succinct than reducing a fraction. When writing a rate, list the single number representing the unit rate, and follow the quantity with a "fraction" composed of the appropriate units of measure.

Speed and Mileage

We are all familiar with speed. When we are driving down a road, our speed is measured in miles per hour. Likewise, as the price of gasoline steadily increases, the public seems to be enamored with cars that have exceptional mileage ratings. These ratings are perfect examples of unit rates. A car that gets 24 miles/gallon (mpg) can go 24 miles on one gallon of gas.

Example 3: John travels 290 miles on 13 gallons of gas. To the nearest whole number, what is the mileage rating for his car?

For the fraction, the miles go in the numerator and the gallons used go in the denominator.

290 miles/13 gallons

$290/13 \approx 22.3$, or 22 rounded to the whole number.

John's car gets 22 mpg.

For unit rates, it is very important to understand the nature of the unit of measure. The answer to the previous example was in "miles per gallon." If we made a mistake and called the unit "gallon per mile," a rating of 22 gallons per mile would be horrible. Would you like a car that use 22 gallons of gas for every mile you drove?

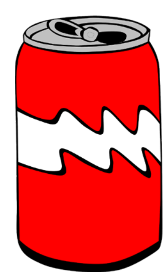
It is possible to reciprocate the entire rate and still have it be correct. 13 gallons/290 miles give a rate of 0.0448 gallons per mile. In this case, however, 0.0448 gallons per mile, although accurate, is not very useful.

Unit Pricing

The **unit price** of an item (or items) is the price per desired unit of measure and is typically used to compare two or more quantities of the same item to see which is the better deal. Unit prices are stated in dollars per unit and are usually rounded to the nearest thousandth. If the unit prices are all less than $1, we state them in cents, rounded to the tenth of a cent.

COMMON MISTAKE

Unit prices are computed as cost per unit of measure, not units per cost.



Example 4: A 6-pack of soda sells for $2.35, and a 12-pack of the same soda sells for $4.75. Determine the unit price of each quantity, and state which one is the better deal.

For the 6-pack, the unit price is $2.35/6 cans ≈ $0.392/can = 39.2¢/can

For the 12-pack, the unit price is $4.75/12 cans ≈ $0.396/can = 39.6¢/can

With the lower unit price, the 6-pack is a better deal. In other words, if you want 12 cans, it would cost less money to buy two 6-packs.

Example 5: A 16-oz can of ravioli costs \$1.88, and a 26-oz can costs \$2.64. Determine the unit price of each quantity, and state which one is the better deal.

For the 16-oz can, the unit price is \$1.88/16 oz ≈ \$0.118/oz = 11.8 ¢/oz

For the 26-oz can, the unit price is \$2.64/26 oz ≈ \$0.102/oz = 10.2 ¢/oz

With the lower unit price, the 26-oz can is a better deal.

Proportions

A **proportion** is a statement that two ratios (or rates) are equal. With the introduction of an equals sign, we can also solve for a missing measure, provided we have the other three measures.

Example 6: Solve for x.

$$\frac{4}{8} = \frac{x}{12}$$

Cross-multiply and solve for x.

8x = 4(12)

x = 48/8 = 6

Example 7: Solve for x.

$$\frac{x}{35} = \frac{9}{12}$$

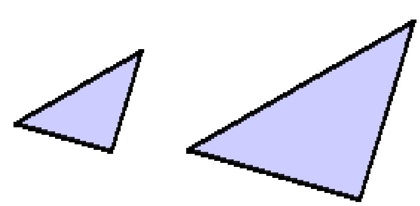
Cross-multiply and solve for x.

12x = 9(35)

x = 26.25

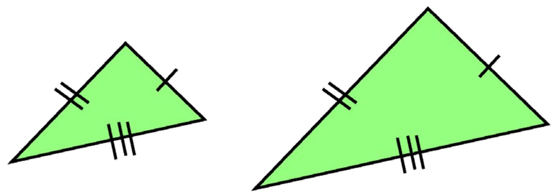
Similar Triangles

Similar triangles are triangles whose angles have the same measure, but their sides have different lengths. The triangles will look identical, but one will be smaller than the other. If two triangles are similar, their side lengths are proportional to each other.

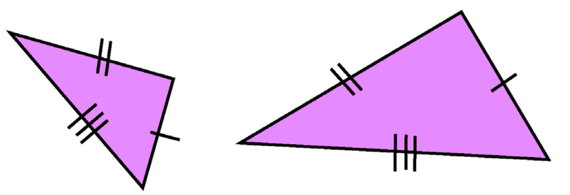


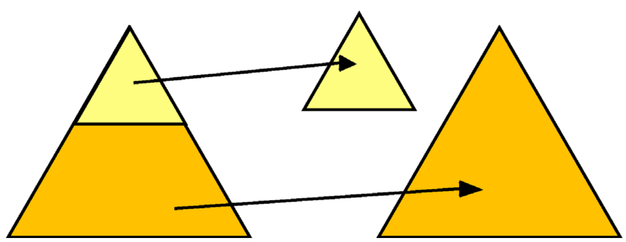
Using the proportionality of the sides, similar triangles are a simple and very useful problem-solving tool. Using the correspondence of the angles of similar triangles, we can find the missing side lengths.

If the triangles have different orientations, make sure you correctly match up the corresponding sides. To help with this, sometimes we put a pair of single tick marks on the first set of corresponding sides, a pair of double tick marks on the second set of corresponding sides, and a pair of triple tick marks on the third set of corresponding sides.

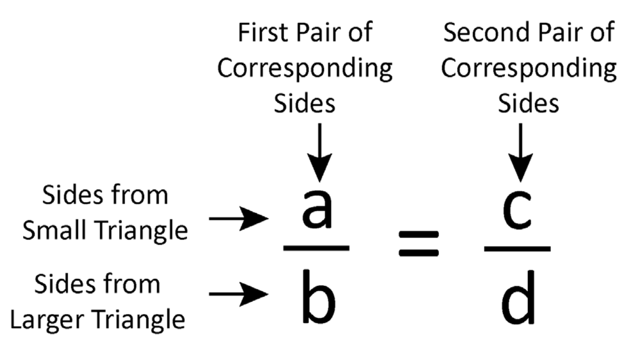


Be careful. Sometimes corresponding triangles may be a little tricky to spot. They may be rotated or even nested.

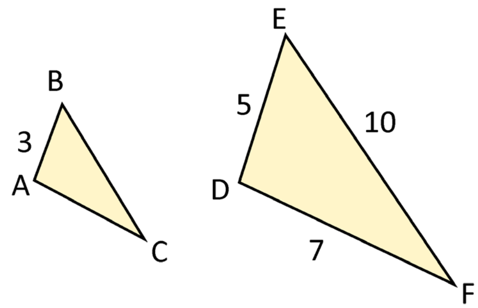




When setting up a proportion, each ratio in the proportion contains a pair of corresponding sides. Furthermore, we must be consistent with which triangle measures we place in the numerators and which ones we place in the denominators of the fractions. That is, think of the following layout when setting up a proportion.



Example 7: Given the following two triangles that are similar, we can see that angle A corresponds to angle D, angle B corresponds to angle E, and angle C corresponds to angle F. Find the lengths of BC and AC, to the nearest hundredth.



We can set up a proportion to find side BC: $\frac{AB}{DE} = \frac{BC}{EF}$

Plugging in the known numbers, we have: $\frac{3}{5} = \frac{BC}{10}$

Cross-multiplying gives us: 5(BC) = 30, which means BC = 6

Similarly, we can set up a proportion to find side AC: $\frac{AB}{DE} = \frac{AC}{DF}$

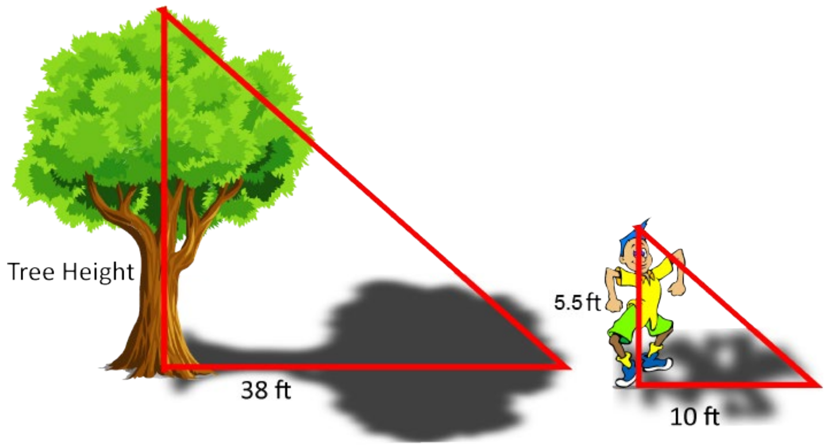
Plugging in the known numbers, we have: $\frac{3}{5} = \frac{AC}{7}$

Cross-multiplying gives us: 5(AC) = 21, so the side AC = 21/5 = 4.2

Example 8: Let's say you are 5.5 feet tall, and, at a certain time of day, you find your shadow to be 10 feet long. At the same time of day, you measure the shadow of a tree to be 38 feet long. TO the nearest foot, how tall is the tree?

Example 8 Continued

Draw the corresponding triangles on the figure and label the sides we know.



Then, set up a proportion that looks like:

$$\frac{Your\ Height}{Tree\ Height} = \frac{Your\ Shadow}{Tree\ Shadow}$$

Plugging in the numbers, we have:

$$\frac{5.5\ ft}{Tree\ Height} = \frac{10\ ft}{38\ ft}$$

That gives us: Tree Height = (38 ft)(5.5 ft)/(10 ft) = 20.9 ft.

By the way, this is not 20 ft 9 inches! 0.9 is 9/10 of a foot. 0.9×12 = 10.8 - which is almost 11 inches. The tree is almost 20 feet 11 inches tall.

Finally, the most common mistake made when working with proportions is the lack of consistency. That is, when setting up the proportion, we have to be sure to numbers from the smaller triangle in the numerators of the ratios, and the numbers from the larger triangle in the denominators. Additionally, we need to make sure the numbers for the corresponding sides are in the same ratios.

COMMON MISTAKE

One of the most common mistakes in working with proportions is putting the numbers in the wrong place in the ratios.

Section 2.4 Exercises

1. What is the ratio of 18 feet to 40 feet. Write the ratio as a fraction in lowest terms.

2. What is the ratio of 32 inches to 14 inches. Write the ratio as a fraction in lowest terms.

3. A school has an enrollment of 13,440 students and a total of 240 faculty members. What is the school's student to teacher ratio? Reduce to lowest terms and write the ratio as two numbers separated by a colon.

4. A school has approximately 800 male and 1000 female students. What is the male to female ratio at the school? Reduce to lowest terms and write the ratio as two numbers separated by a colon.

5. Sherry's weekly gross pay is $285, and she has $80 worth of deductions made from her paycheck. What is the ratio of her take-home pay to her gross pay? Reduce to lowest terms and write the ratio as two numbers separated by a colon.

6. Joe travels 180 miles on 5 gallons of gas. To the nearest whole number, what is the mileage rating for his car?

7. Julie travels 315 miles in 8 hours. To the nearest tenth, what is her average speed?

8. Juan packs 286 books into 11 identical boxes. What is the rate for books per box?

9. A 16-oz bottle of olive oil costs \$3.38, and a 24-oz bottle costs \$4.64. Determine the unit price of each quantity, and state which one is the better deal.

10. A 12-oz box of cereal costs \$2.18, and a 22-oz box costs \$3.14. Determine the unit price of each quantity, and state which one is the better deal.

11. A 4-pack of paper towels costs \$2.24, and a 6-pack of the same towels costs \$3.48. Determine the unit price of each quantity, and state which one is the better deal.

12. A 10-oz tube of toothpaste costs 98¢, and a 22-oz tube costs \$2.04. Determine the unit price of each quantity, and state which one is the better deal.

13. Solve the proportion for the given variable. Round to the nearest tenth, if necessary. $\frac{14}{8} = \frac{12}{x}$

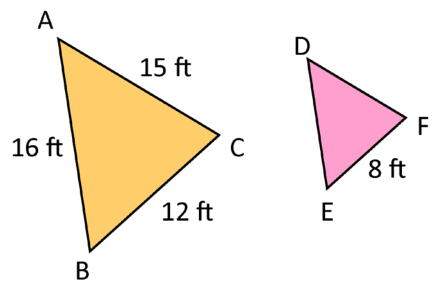
14. Solve the proportion for the given variable. Round to the nearest tenth, if necessary. $\frac{7}{9} = \frac{x}{12}$

15. Solve the proportion for the given variable. Round to the nearest tenth, if necessary.
$$\frac{24}{39} = \frac{y}{15}$$

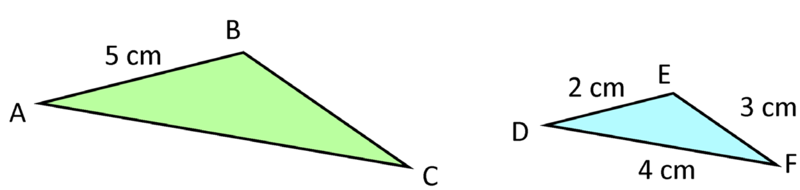
16. Solve the proportion for the given variable. Round to the nearest tenth, if necessary.
$$\frac{4}{c} = \frac{5}{12}$$

17. Solve the proportion for the given variable. Round to the nearest tenth, if necessary.
$$\frac{15}{8} = \frac{6}{n}$$

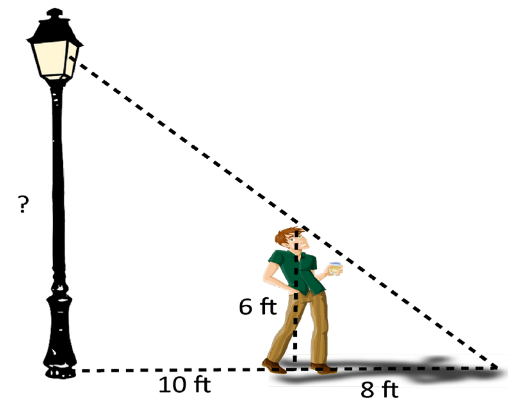
18. Given that the pictured triangles are similar, find the lengths sides DE and DF. Round your answers to the nearest tenth, as necessary.



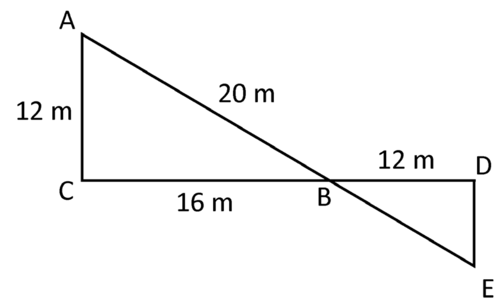
19. Given that the pictured triangles are similar, find the lengths sides AC and BC. Round your answers to the nearest tenth, as necessary.



20. While standing 10 ft away from a light pole, Billy notices his shadow is 8 feet long. If Billy is 6 feet tall, how tall is the light pole?



21. In the given figure, triangle ABC is similar to triangle EBD. What are the measures of BE and DE?



Section 2.4 Exercise Solutions

1. 18/40 = 9/20

2. 32/14 = 16/7

3. 13440/240 = 56/1

4. 8/10 = 4/5 → 4:5

5. First, we have to find the take-home pay, which is the gross pay less the deductions. $285 - $80 = $205. That makes the ratio $205/$285 = 41/57 → 41:57

6. 180 mi/5 gal = 180/5 mi/gal = 36 mpg

7. 315 mi/8 hr = 315/8 mi/hr = 39.4 mph

8. 286 books/11 boxes = 286/11 books/box = 26 books/box

9. The 16-oz bottle is $0.211/oz, and the 24-oz bottle is $0.193/oz. The 24-oz bottle is a better deal.

10. The 12-oz box is $0.182/oz, and the 22-oz box is $0.143/oz. The 22-oz box is a better deal.

11. The 4-pack is 56 ¢/roll, and the 6-pack is 58 ¢/roll. The 4-pack is a better deal.

12. Be sure to change the 98 to $0.98 before you do the unit price for the 10-oz tube. The 10-oz tube is $0.098/oz, and the 22-oz tube is $0.093/oz. The 22-oz tube is a better deal.

13. 14x = 96 → x = 96/14 = 6.857… → 6.9 to the tenth

14. 9x = 84 → x = 84/9 = 9.333… → 9.3 to the tenth

15. 39y = 360 → y = 360/39 = 9.2307… → y = 9.2 to the tenth

16. 5c = 48 → c = 48/5 = 9.6

17. 15n = 48 → n = 48/15 = 3.2

18. For DE: DE/AB = EF/BC → DE/16 = 8/12 → DE = 128/12 = 12.666… → DE = 10.7 ft
    For DF, DF/AC = EF/BC → DF/15 = 8/12 → DF = 120/12 = 10 → DF = 10 ft

19. For AC: DE/AB = DF/AC → 2/5 = 4/AC → AC = 20/2 = 10 → AC = 10 cm
    For BC, DE/AB = EF/BC → 2/5 = 3/BC → BC = 15/2 = 7.5 → BC = 7.5 cm

20. Be careful. The base of the larger triangle is 18 feet, not just 10 feet.
    Billy/Lamp = Billy's Shadow/Lamp Projection → 6/x = 8/18 → x = 108/8 = 13.5
    The lamppost is 13.5 ft high.

21. DE/AC = DB/CB → DE/12 = 12/16 → DE = 144/16 → DE = 9 m
    BE/BA = DB/CB → BE/20 = 12/16 → BE = 240/16 → BE = 15 m

2.5: RIGHT TRIANGLES, DISTANCE & SLOPE

Right Triangles

Right triangles are some of the most useful figures in society. We will focus on using the side lengths to calculate distance and slope. Beyond that, an entire branch of mathematics, trigonometry is based off other relationships between the side lengths and angle measures.

Square Roots

Given a quantity, the **principal square root** of that quantity is the positive number we must square to give that quantity. For our purposes, we will refer to the "principal square root" as just the "square root." For an example, 5 is the square root of 25 because $5^2 = 25$. For notation, we write $\sqrt{25} = 5$.

For numbers like 25, 36, 81, and 100, finding the square root is easily done by inspection. These numbers are **perfect squares**. How do we find the square root of a number that is not a perfect square? Although there are ways to do this by hand, the fastest way is to just use a calculator. Furthermore, unless directed otherwise, we will round square roots to the nearest hundredth.

The Pythagorean Theorem

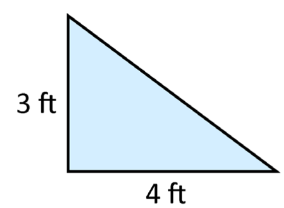
The longest side of a right triangle is always the side opposite the right angle and is called the **hypotenuse**. The other two sides are referred to as **legs**. The **Pythagorean Theorem** states: **In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.** That is, hypotenuse^2 = leg^2 + leg^2. The more colloquial version of this is $a^2 + b^2 = c^2$, where c is always the length of the hypotenuse.

The Pythagorean Theorem

In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

That is, $a^2 + b^2 = c^2$, where a and b are the side lengths, and c is the length of the hypotenuse.

Example 1: Find the length of the hypotenuse in the right triangle.



Since the legs measure 3 ft and 4 ft, the Pythagorean Theorem states:
(hypotenuse)^2 = (3 ft)^2 + (4 ft)^2
(hypotenuse)^2 = 9 ft^2 + 16 ft^2
(hypotenuse)^2 = 25 ft^2
hypotenuse = 5 ft

COMMON MISTAKE

Some students always assume the given side lengths are always the legs, and then solve for the hypotenuse. In some cases, a given measure may be the hypotenuse.

Example 2: Find the length of the missing side in a right triangle if the hypotenuse is 6 ft and one of the sides is 2 feet long. Round your answer to the nearest hundredth of a foot.

We are given one of the sides and the hypotenuse, so, using $a^2 + b^2 = c^2$:
$(2 \text{ ft})^2 + (\text{side})^2 = (6 \text{ ft})^2$
$4 \text{ ft}^2 + (\text{side})^2 = 36 \text{ ft}^2$
$(\text{side})^2 = 36 \text{ ft}^2 – 4 \text{ ft}^2$
Thus, side = 5.6568… ft → 5.66 ft

Example 3: Find the length of the hypotenuse in a right triangle if the side lengths are 60 cm and 11 cm. Round your answer to the nearest hundredth of a cm.

$a^2 + b^2 = c^2$
$(60 \text{ cm})^2 + (11 \text{ cm})^2 = c^2$
$c^2 = 3600 \text{ cm}^2 + 121 \text{ cm}^2 = 3721 \text{ cm}^2$
$c = \sqrt{3721\ cm^2} = 61$ cm

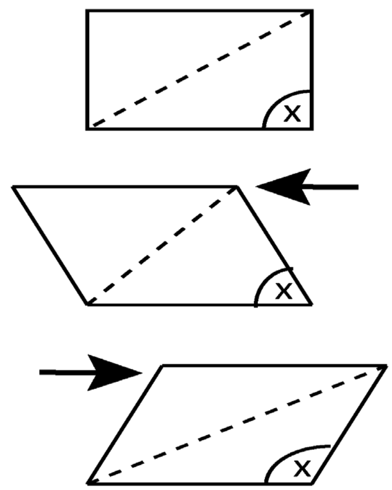
Example 4: Find the length of the hypotenuse in a right triangle if both sides are 13 m. Round your answer to the nearest hundredth of a m.

$a^2 + b^2 = c^2$
$(13 \text{ m})^2 + (13 \text{ m})^2 = c^2$
$c^2 = 169 \text{ m}^2 + 169 \text{ m}^2 = 338 \text{ m}^2$
$c = \sqrt{338\ m^2} = 18.38477...$ m → 18.38 m

Remember, the hypotenuse is ALWAYS the longest side in the right triangle.
Additionally, it is ALWAYS the side opposite the right angle.

What good is the Pythagorean Theorem, anyway?

Carpenters still use the 3-4-5 triangle to square corners. To do this, we need to think of a window frame and what happens to that frame when we anchor its bottom, but push the top to the left or the right.



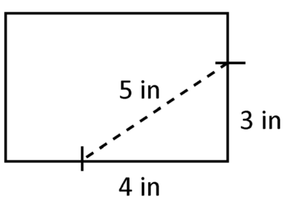
In the figures on the right, a frame is shown in three states. The top frame is considered "square" because the angle x is a right angle.

In the middle frame, the top has been pushed to the left. Notice the angle x is less than 90° and the dashed line is shorter than the same line in the top frame.

In the bottom frame, the top has been pushed to the right. Notice the angle x is greater than 90° and the dashed line is longer than the same line in the top frame.

When the bottom of the window frame is nailed down, we can adjust the length of the dashed line by pushing the top of the frame to the left or right.

We can be sure the window frame is "square" by a simple application of the Pythagorean Theorem. From a corner, we can make marks on the frame that are 3 inches in one direction and 4 inches in the other. Then, the whole frame can be adjusted until the straight-line distance between the two marks is 5 inches.

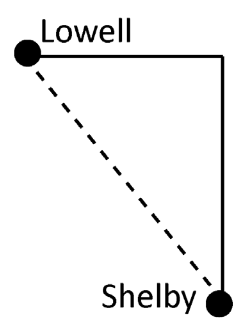


Since a triangle with a side ratio of 3-4-5 obeys the Pythagorean Theorem (that is, $3^2 + 4^2 = 5^2$), then that triangle *has* to be a right triangle. In other words, if you force one of these 3-4-5 triangles into a corner of a frame, then, according to the Pythagorean Theorem, the angle across from the hypotenuse (the side of length 5) has to be a right angle.

Distance

We have all heard the phrase "The shortest **distance** between any two points is a straight line." To find that distance, we use the Pythagorean theorem.

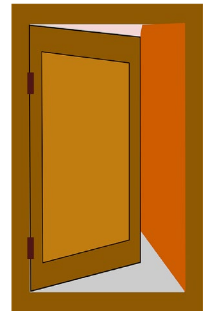
Example 5: To travel from Lowell to Shelby, Tricia drives 6 miles east and then 8 miles south. What is the straight-line distance from Lowell to Shelby?



$a^2 + b^2 = c^2$
$(6\text{ mi})^2 + (8\text{ mi})^2 = c^2$
$36\text{ mi}^2 + 64\text{ mi}^2 = c^2$
$c^2 = 100\text{ mi}^2 \rightarrow c = 10\text{ mi}$

The straight-line distance from Lowell to Shelby is 10 miles.

Example 6: A door frame measures 36 in wide by 80 in tall. What is the length of the diagonal, to the nearest quarter inch?



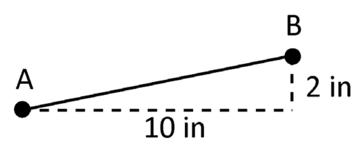
$a^2 + b^2 = c^2$
$(36\text{ in})^2 + (80\text{ in})^2 = c^2$
$1296\text{ in}^2 + 6400\text{ in}^2 = c^2$
$c^2 = 7696\text{ in}^2 \rightarrow c = 87.7268\ldots\text{ in} \rightarrow 87.75\text{ in}$

The diagonal measure is 87 ¾ in.

Slope

Another application of right triangles involved slope. The **slope** of a line is defined as the ratio of its vertical change to its horizontal change and is commonly represented by the lower-case letter m. Alternatively, the slope of a line can be determined by finding the fraction of as the rise over the run. Do realize, however, since slope is a ratio, it should contain two numbers. Keep in mind, if the slope is a whole number, such as m = 5, the ratio of the rise to the run is 5:1.

Example 7: Find the slope of the line from point A to point B.



Slope = Rise/Run = 2 in /10 in = 1/5

Positive Slope, Negative Slope, Zero Slope and Undefined Slope

Technically, if a line is slanted upward going from left to right, like a road going up a hill, the slope is positive. Alternatively, it the line is falling, like a road going downhill, the slope is negative. That said, whether you are going up the hill or down the hill is a matter of perspective. So, for our purposes, we will simply state slopes as positive numbers.

A horizontal line has a slope of 0, and a vertical line has no slope. Think about those two for a moment. Slope is the indication of the steepness of a line. If the line were a road, a horizontal line would represent a flat road with an incline of 0, as 0 is the number between all the positive and all the negative numbers. If, however, the road was vertical - commonly called a wall ☺ - no one is going to drive down that "road." Hence, the slope is undefined.

Applications of Slope

When describing roads, slopes are referred to as **grades** and are usually stated as percents. Percent literally means "per 100." So, a sign indicating an 18% grade has a rise of 18 feet for every 100 feet of run. For example, if a survey crew determines the road rises 180 feet for every 1000 feet it runs, the slope of the road is 180/1000, which is a grade of 18%.

Example 8: A survey crew using a 5-foot high stand, sights a spot on a road that is 62 feet away What is the grade of the road, to the nearest whole percent?

The rise is the height of the measuring stand and the run is the distance to the spot on the road.

Slope = Rise/Run = 5ft/62 ft = 0.0806…

Change this to a percent by moving the decimal point two places, and then rounding to the nearest whole percent.

The grade of the road is 8%.

Measuring road grades is important in mountainous states. Installed along a steep downhill grade of a main road, **runaway truck ramps** allow for large vehicles experiencing braking problems to safely stop without a violent crash.

For another use of slope, a building code indicates asphalt roofing shingles are not to be used on a roof that has very little steepness, which is referred to as the **pitch** of the roof.

The minimum pitch must be a rise of 1 foot for every 3 feet of horizontal distance (run) covered. That means the steepness (slope) of the roof must be greater than or equal to 1/3. If a roofer measures the roof and finds there are 10 feet of rise for a horizontal run of 20 feet, the steepness of the roof is 1/2, which is greater than 1/3, and, hence, acceptable for asphalt shingles.

Section 2.5 Exercises

1. Find the following. Round to the nearest hundredth, as necessary.
   a. $\sqrt{36}$ b. $\sqrt{121}$ c. $\sqrt{400}$

2. Find the following. Round to the nearest hundredth, as necessary.
   a. $\sqrt{37}$ b. $\sqrt{190}$ c. $\sqrt{63}$

3. If a right triangle has legs measuring 6 feet and 8 feet, how long is the hypotenuse? Round your answer to the nearest whole foot.

4. If a right triangle has legs measuring 16 feet and 8 feet, how long is the hypotenuse? Round your answer to the nearest tenth of a foot.

5. If a right triangle has one leg that measures 12 cm, and a hypotenuse that measures 13 cm, how long is the other side? Round your answer to the nearest tenth of a cm.

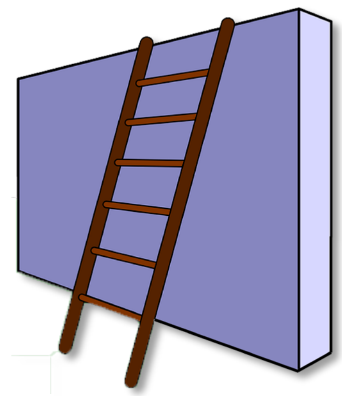
6. If a right triangle has one leg that measures 13 cm, and a hypotenuse that measures 22 cm, how long is the other side? Round your answer to the nearest hundredth of a cm.

7. If both legs of a right triangle measure 6 inches, how long is the hypotenuse? Round your answer to the nearest tenth of an inch.

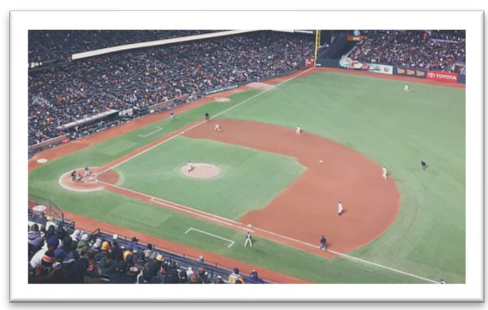
8. If you drive 12 miles north, make a right turn and drive 9 miles east, how far are you, in a straight line, from your starting point? Round your answer to the nearest whole mile.

9. If you drive 100 miles east, make a right turn and drive 59 miles south, how far are you, in a straight line from your starting point? Round your answer to the nearest tenth of a mile.

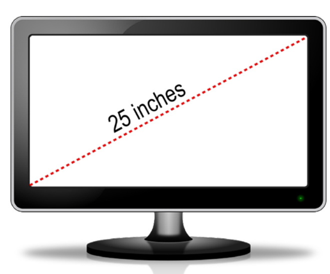
10. If a 10-foot ladder is leaned against the top of a 9-foot high wall, how far will the base of the ladder be from the bottom of the wall? Round your answer to the nearest tenth of a foot.



11. On a baseball diamond there are 90 feet between the bases, and the base paths are at right angles. What is the straight-line distance from home plate to second base, to the nearest whole foot?



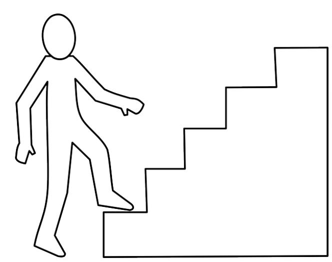
12. In a 25-inch television, the length of the screen's diagonal is 25 inches. If the screen's height is 15 inches, what is the width? Round your answer to the nearest whole inch.



13. A survey crew worker uses a site instrument that is on a 5-ft stand. He sights a spot on the road 42 feet away. What is the slope of the road? What is the grade, to the nearest whole percent?

14. If the grade of a road is consistent 5%, how far do you need to travel to increase your elevation by 300 ft? Round your answer to the nearest whole foot.

15. A contractor installing a staircase in a new house intends to follow the "7-11 Rule," which states there is a 7-in rise for every 11-in run for each stair step. What will be the slope of the staircase? Also, if the staircase must rise 10 feet, what will be the total horizontal length needed for the staircase? Round your answer to the nearest tenth of a foot.



16. A construction worker needs to raise some materials to the top of a 25-ft tall building by using a computer-controlled crane. If the materials are 20 feet away from the base of the building, what slope needs to be programmed into the computer to accomplish the delivery?

Section 2.5 Exercise Solutions

1. a. 6 b. 11 c. 20

2. Use a calculator. a. 6.08 b. 13.78 c. 7.94

3. $a^2 + b^2 = c^2 \rightarrow (6\text{ ft})^2 + (8\text{ ft})^2 = c^2 \rightarrow c^2 = 100\text{ ft}^2 \rightarrow c = 10\text{ ft}$

4. $a^2 + b^2 = c^2 \rightarrow (16\text{ ft})^2 + (8\text{ ft})^2 = c^2 \rightarrow c^2 = 320\text{ ft}^2 \rightarrow c = 17.8885\ldots\text{ ft} \rightarrow 17.9\text{ ft}$

5. $a^2 + b^2 = c^2 \rightarrow (12\text{ cm})^2 + b^2 = (13\text{ cm})^2 \rightarrow b^2 = 25\text{ cm}^2 \rightarrow b = 5\text{ cm}$

6. $a^2 + b^2 = c^2 \rightarrow (13\text{ cm})^2 + b^2 = (22\text{ cm})^2 \rightarrow b^2 = 315\text{ cm}^2 \rightarrow b = 17.7482\ldots\text{ cm} \rightarrow 17.75\text{ cm}$

7. $a^2 + b^2 = c^2 \rightarrow (6\text{ in})^2 + (6\text{ in})^2 = c^2 \rightarrow c^2 = 72\text{ in}^2 \rightarrow c = 8.4852\ldots\text{ in} \rightarrow 8.5\text{ in}$

8. $a^2 + b^2 = c^2 \rightarrow (12\text{ mi})^2 + (9\text{ mi})^2 = c^2 \rightarrow c^2 = 225\text{ mi}^2 \rightarrow c = 15\text{ mi}$

9. $a^2 + b^2 = c^2 \rightarrow (100\text{ mi})^2 + (59\text{ mi})^2 = c^2 \rightarrow c^2 = 13{,}481\text{ mi}^2 \rightarrow c = 116.1077\ldots\text{ mi} \rightarrow 116.1\text{ mi}$

10. $a^2 + b^2 = c^2 \rightarrow a^2 + (9\text{ ft})^2 = (10\text{ ft})^2 \rightarrow a^2 = 19\text{ ft}^2 \rightarrow a = 4.3588\ldots\text{ ft} \rightarrow 4.4\text{ ft}$

11. $a^2 + b^2 = c^2 \rightarrow (90\text{ ft})^2 + (90\text{ ft})^2 = c^2 \rightarrow c^2 = 16{,}200\text{ ft}^2 \rightarrow c = 127.279\ldots\text{ ft} \rightarrow 127\text{ ft}$

12. $a^2 + b^2 = c^2 \rightarrow a^2 + (15\text{ in})^2 = (25\text{ in})^2 \rightarrow a^2 = 400\text{ in}^2 \rightarrow a = 20\text{ in}$

13. Rise/Run = 5 ft/42 ft = 0.11904… = 11.904…% → 12%

14. The slope is 5%, which is 5/100. Then, rise/run = 300 ft/run = 5/100.
    5(run) = 100(300 ft) → run = 6000 ft

15. Slope = rise/run = 7 in/11 in = 7/11
    7/11 = 10 ft/run → 7(run) = (10)(11 ft) → run = 15.714… ft → 15.7 ft

16. Slope = rise/run = 25 ft/20 ft = 5/4

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CHAPTER 3: CONSUMER MATH

Money. It controls nearly everything in the world. Those who have it are often seen as more powerful and influential than those who do not. So, having a basic understanding of how money grows and is (or should be) spent is essential.

Understanding percentages is key to understanding how money works. Budgets are based on different percents of available money being allocated in specific ways. Sale discounts and taxes are based on the percent of cost of the items at hand. Credit cards and loans operate on interest rates, which – you guessed it – are given in percents.

The single largest purchase most people will make in a lifetime is a home. A basic understanding of mortgages and finance charges will make you a much wiser homebuyer and could easily save you tens of thousands of dollars over a 20 to 30-year period.

We could go on, but you get the picture.

CHAPTER OUTLINE AND OBJECTIVES

Section 3.1: Percent

A. Be able to identify amounts, bases and percents, and solve percent problems.
B. Be able to compute the commission and tips.
C. Be able to compute discounts, sales taxes, rebates & price markups.
D. Be able to create a simple monthly budget.

Section 3.2: Simple & Compound Interest

A. Be able to compute simple interest.
B. Be able to compute compounded amounts.
C. Be able to correctly use the exponentiation feature of your calculator.

Section 3.3: Installment Buying & Credit Cards

A. Understand installment buying.
B. Be able to compute finance charges & monthly payments.
C. Be able to compute credit card balances.

Section 3.4: Leases

A. Understand the various components of a lease.
B. Be able to compute and compare different leases.
C. Be able to determine if it is better to lease or buy something.

Section 3.5: Mortgages

A. Be able to perform amortization calculations using tables, spreadsheets & calculators
B. Be able to compute monthly mortgage payments.
C. Understand mortgage affordability guidelines.

3.1: PERCENT

When finding a **percent** of an amount or an amount that is a percent of a different quantity, most people know they either need to multiply or divide by the decimal value of the percent. Unfortunately, most people can't quite remember when to multiply and when to divide. Fortunately, we can reduce all of those problems to the same equation and let our algebra skills tell us what to do.

Identifying the Parts in a Percent Problem

In simple percent problems, there is a percent, a base, and an amount. The numeric value of the percent, p is easy to spot - it has a % immediately after it, and for calculation purposes, the percent must be changed to its decimal equivalent. The **base**, b is the initial quantity and is associated with the word "of." The **amount**, a is the part being compared with the initial quantity and is associated with the word "is."

Example 1: Identify the amount, base, and percent in the following statement.

65% of 820 is 533.

Percent = 65%, Base = 820, Amount = 533

Example 2: Identify the amount, base, and percent in the following statement. If a specific value is unknown, use the variable p, b, and a for the percent, base, and amount, respectively. Do not solve the statement; just identify the parts.

What is 35% of 95?

Percent = 35%, Base = 95, Amount = a

Solving Percent Problems Using an Equation

Always begin by identifying the percent, base, and amount in the problem, and remember, usually, one of them is unknown. If we are given a percent in the problem, change it to a decimal for the calculation. If we are finding a percent, the calculation will result in a decimal and we must change it to a percent for our answer.

Once we have identified the parts, we then need to recognize the basic form of the **percent equation**. An amount, a, is some percent, p, of a base, b. In other words, **a = pb**, where p is the decimal form of the percent.

Percent Equation

a = pb, where p is the decimal form of the percent.

When setting up the equation, if the a is the only unknown, find its value by simplifying the other side of the equation (by multiplying the values of p and b). If the unknown quantity is b, divide both sides of the equation by the value for p. Likewise, if the unknown quantity is p, divide both sides of the equation by the value for b.

Example 3: What is 9% of 65?

The percent is 9%, so we will use p = 0.09. 65 is the base, so b = 65. Thus, the unknown quantity is a.

a = 0.09(65) = 5.85

Example 4: 72 is what percent of 900?

900 is the base, and 72 is the amount. The unknown quantity is the percent.

72 = p(900)
72/900 = p

So, p = 0.08, which is 8%.

Example 5: 150 is what percent of 40?

40 is the base, and 150 is the amount. The unknown quantity is the percent.

150 = p(40)
150/40 = p

So, p = 3.75, which is 375%.

General Applied Problems Involving Percents

To solve any application problem involving a percent, a good strategy is to rewrite the problem into the form "a is p% of b." Remember, we should always begin by clearly identifying the amount, base and percent.

Example 6: Of the 130 flights at Orange County Airport yesterday, only 105 of them were on time. What percent of the flights were on time? Round to the nearest tenth of a percent.

First, identify the base is 130 flights, and 105 is the amount we are comparing to the base. The percent is the unknown.

Then, we should recognize the simplified form of this question is "105 is what percent of 130?" This gives us the equation: 105 = p(130).

105 = p(130)
105/130 = p = 0.8076923… = 80.76923…%

Rounding to the tenth of a percent, we state, "80.8% of the flights were on time."

Commissions and Tips

One of the most fundamental and direct applications of percent is calculating **commissions** and **tips**. Many salespeople work on commission. That is, they get paid a percent of the items they sell.

Example 7: Al sells cars. If he earns 25% of the price on each car he sells, how much does he make when he sells a $15,000 car?

The base is $15,000 and the percent is 25%. His commission is the unknown amount, a.
a = 0.25($15,000) = $3750.

Al's commission on a $15,000 car is $3750.

Tips are a way of life for many workers in the service industry. As a rule of thumb, if you are dining out, you should tip your server 10-20% of the cost of the meal (before taxes). 10% is really easy to calculate; just move the decimal point one place. Then, for 20%, we can find 10% and double it.

Example 8: After enjoying a nice meal, Penny and Leonard are presented with a check for $54.30. They felt the service was exceptional and want to leave a 20% tip. How much is the tip?

10% is $5.43. Double it. That makes 20% = $10.86.

Sales Tax and Discounts

Sales taxes are computed as a percent of the cost of a taxable item. Then, when we find the dollar amount of the tax, we add it to the cost. **Discounts** are also computed as a percent of the original price of an item. For discounts, however, when we find the dollar amount of the discount, we deduct it from the original price. If both a discount and sales tax are involved in the same computation, we need to know which one gets applied first. In most cases, the sales tax gets computed on the discounted price. However, in some cases, we may get a discount, but will still be responsible for the taxes based on the original price – it all depends on the wording used in the situation. The later of those two situations is usually referred to as a **rebate**. Often, rebates are given as dollar amounts, instead of percents.

Next, we need to make sure we answer the question being asked. If the question asks for the tax, we should ONLY give the tax as our answer. Likewise, if the question asks for the discount, we should ONLY give the amount of the discount as our answer. If the question asks for a sales price, then we need to be sure to subtract the discount from the posted price. If the question asks for the total cost of an object, we need to be sure to add the tax to the item's price.

Finally, when working with money, unless directed otherwise, we should always round to the nearest cent. Think about it. When we fill our gas tank and the total is $23.01, we have to pay that penny. Likewise, if your paycheck is for $543.13, you don't just get the $543.

Unless specifically directed to do otherwise,
ALWAYS round money answers to the nearest cent.

COMMON MISTAKE

"Round to the nearest cent" means to round to the hundredths place. Some students incorrectly round to the tenth with directed to round to the nearest cent. Remember, when directed to round to the cent, we must display our answers with two decimal places.

Example 8: Find the total cost of a $125 TV if the sales tax rate is 7.25%.

The base cost of the TV is $125, so b = 125. As a decimal, p = 0.0725.
a = 0.0725(125) = 9.0625. So, to the cent, the sales tax is $9.06.

That makes the total cost of the TV $125 + $9.06 = $134.06.

Example 9: A pair of shoes is on sale for 20% off. If the original price of the shoes was \$65, what is the sale price?

The base cost of the shoes is \$65, so b = 65. p = 0.20.
a = 0.2(65) = 13. So, the discount is \$13.

This makes the sale price of the shoes \$65 - \$13 = \$52.

Pay attention to the question that gets asked. Some problems may ask for the sales tax, and others may ask for the total cost. A very common mistake with percent problems is answering the wrong question.

Making a Budget

A **budget** is a plan of how we will spend our money each month. Without a budget, we might run out of money before our next paycheck (assuming we get paid monthly). The first step in creating a monthly budget is to determine our net income (after taxes and other paycheck deductions). From there, we need to look at how we spend our money each month.

Although it is possible to create many specific categories, we will lump our expenses into the following:

1. Housing – This is not limited to just rent or a mortgage payment. It includes other expenses related to housing, such as property taxes, insurance, maintenance and repairs. In general, this should be limited to 25-40% of our net income. If our car is paid off and we do not have student loan payments, we can spend more on housing. If we have those extra monthly loan payments, expect to limit this category to closer to 25-30%.

2. Utilities – Water, electricity, gas, trash, cell phone, cable, Internet and other monthly services. The total of all these utility bills should not be more than 10% of our net income.

3. Groceries – Typically, 10-15% of our net income should be devoted to groceries, but it can vary greatly depending on the size of our family.

4. Personal Care – Like groceries, this category can be pretty volatile, depending on your family situation. Generally, young, single people have lower health care expenses, but this category also includes hair care and gym memberships. After all, we do need to take care of ourselves. Ideally, this one should be limited to no more than 15% of our monthly income.

5. Loan Payments – This broad category covers car payments, student loans, and credit card payments on unpaid balances. If we find this category exceeding 10% of our monthly net income, we may have a problem.

6. Savings/Emergency Fund – We should make every effort to set aside 10% of our net income each month. Having that money set aside for emergency airfare or an unexpected car repair can make life a little less stressful.

7. Entertainment – This is the category that is easiest to cut. Buying that coffee on the way to work, hitting the club in Friday nights, going to the movies or a concert, and even dining out. Limiting entertainment expenses to 5 to 10% of our income is a good start. However, some quick math indicates, by this point, we may have already consumed 100% of our monthly net income. If so, our entertainment expenses can simply be the amount we saved from our expenses in the previous month.

Since housing expenses are the most important and consistent of the monthly expenses, we start our budget by finding the percent of our income devoted to that category. Then, we can split of the remaining income across the other categories according to our desired percents. In the end, we should make sure all the percents, including the one for housing, add up to 100%.

Example 10: Amy has a net monthly income of \$3500 and incurs \$1050 in housing expenses each month. What percent of her monthly budget goes toward housing?

Housing Expenses = What Percent of her Monthly Income?
\$1050 = p(\$3500)
p = \$1050/\$3500 = 0.30 = 30%

Amy's housing expenses account for 30% of her monthly budget.

Example 11: Amy has a net monthly income of \$3500 and is willing to spend 10% of that income on utilities, 12% on personal care, and 7% on entertainment. How much money does she devote to each category?

Utilities = 10% of \$3500
a = 0.10(\$3500) = \$350

Personal Care = 12% of \$3500
a = 0.12(\$3500) = \$420

Entertainment = 7% of \$3500
a = 0.07(\$3500) = \$245

Amy budgets \$350 for utilities, \$420 for personal care, and \$245 for entertainment each month.

Example 12: Ronnie has a net monthly income of \$4200. Last month he spent 28% of his income on housing, 10% on utilities, 8% on groceries, 11% on personal care, 9% for loan payments, and 12% on entertainment. The rest of his monthly income went into savings. How much money did he put into his savings?

He spent 28% + 10% + 8% + 11% + 9% + 19% = 85%
That means he saved 15% of his income.
a = 0.15(\$4200) = \$630

Ronnie put \$630 into savings last month.

Using a Spreadsheet/Calculator

If we have never made a monthly budget before, we may not have a very good idea of the actual percents to use for each category. Thus, we may need to track our spending habits for a couple months.

Collect and save every receipt for a month and sort them into general categories, as outlined earlier in this section. At the end of the month enter the total for each category into a spreadsheet. You may be surprised at how much you spend in each category.

If you divide the money spent for a category by your monthly net income, you can determine the percent needed for that category. If you do this for 2-3 consecutive months, you will not only have a better idea of where your money is going, but you will also gain a new appreciation for expenditure limitations.

Section 3.1 Exercises

For all of these exercises, unless directed otherwise, round bases and amounts to the nearest hundredth, and percents to the nearest tenth of a percent.

1. What is 10% of 58?

2. What is 140% of 35?

3. 90 is what percent of 120?

4. 120 is what percent of 90?

5. 0.13 is what percent of 5?

6. 72 is 37% of what?

7. 45 is 112% of what?

8. 16 is 0.24% of what?

9. A student correctly answered 23 out of 30 questions on a quiz. What percent is this?

10. A basketball player made 13 out of 18 free throw attempts. What percent is this?

11. A recent survey showed 38% of people thought the President was doing a good job. If 5000 people were surveyed, how many people approved of the President's job performance?

12. A bowl of cereal contains 43 grams of carbohydrates, which is 20% of the recommended daily value of a 2000-calorie diet. What is the total number of grams of carbohydrates in that diet?

13. A bowl of cereal contains 190 mg of sodium, which is 8% of the recommended daily value in a 2000-calorie diet. What is the total number of milligrams of sodium in that diet?

14. Wade wants to buy a new computer priced at $1200. If the sales tax rate in his area is 5.75%, what will be the amount of sales tax and total cost for this computer?

15. In the bookstore, a book is listed at $24.95. If the sales tax rate is 6.5%, what will be the amount of sales tax and the total cost of the book?

16. Mike has breakfast at a local restaurant and notices the sub-total for his food and coffee is $8.98. If the sales tax is $0.67, what is the sales tax rate?

17. Nancy wants to buy a sweater that is priced at $45. If the store is running a 40% off sale, what will be the amount of the discount and the sale price of the sweater?

18. Sue finds a CD she likes that normally costs $19.99. If the CD is labeled 75% off, and the sales tax rate is 5.5% (of the sale price), what will be the total cost of the CD?

19. To deal with slumping sales, Eric's boss cut his salary by 15%. If Eric was making $52,000 per year, what will be his new annual salary?

20. John's monthly net salary is $3500, and he budgets $805 for housing expenses. What percent of his net monthly salary goes housing expenses?

21. Jane budgets 8% of her net salary for groceries. If she spent $300 on groceries last month, what is her monthly net salary?

22. Julie has a net monthly income of $4100 and is willing to spend 11% of that income on groceries, and 9% on entertainment. How much money does she devote to each category?

23. Bob has a net monthly income of $3950. Last month he spent 31% of his income on housing, 11% on utilities, 8% on groceries, 5% on personal care, 1% for loan payments, and 10% on entertainment. The rest of his monthly income went into savings. How much money did he put into his savings?

Section 3.1 Exercise Solutions

1. a = pb → a = 0.1(58) → a = 5.8

2. a = pb → a = 1.4(35) → a =49

3. a = pb → 90 = p(120) → p = 90/120 = 0.75 =75%

4. a = pb → 120 = p(90) → p = 120/90 = 1.3333… = 133.333…% → p = 133.3%

5. a = pb → 0.13 = p(5) → p = 0.13/5 = 0.026 = 2.6%

6. a = pb → 72 = 0.37(b) → b = 72/0.37 = 194.59459… → b = 194.59

7. a = pb → 45 = 1.12(b) → b = 45/1.12 = 40.1785… → b = 40.18

8. a = pb → 16 = 0.0024(b) → b = 16/0.0024 = 6666.66666… → b = 6666.67

9. a = pb → 23 = p(30) → p = 23/30 = 0.76666666… = 76.6666…% → p = 76.7%
   The student answered 76.7% of the questions correctly.

10. a = pb → 13 = p(18) → p = 13/18 = 0.722222… = 72.222…% → p = 72.2%
    The player made 72.2% of his free throws.

11. a = pb → a = 0.38(5000) → a = 1900
    1900 of the people surveyed approved of the President’s job performance.

12. a = pb → 43 = 0.2(b) → b = 43/0.2 = 215
    215 carbohydrates are recommended in that diet.

13. a = pb → 190 = 0.08(b) → b = 190/0.08 = 2375
    2375 mg of sodium are recommended in that diet.

14. a = pb → Sales Tax = 0.0575($1200) = $69. Total Cost = Price + Tax → $1200 + $69 = $1269.
    The sales tax is $69, and the total cost of the computer is $1269.

15. a = pb → Sales Tax = 0.065($24.95) = $1.62175 → $1.62. Total Cost = $24.95 + $1.62 = $26.57.
    The sales tax is $1.62, and the total cost is $26.57.

16. a = pb → $0.67 = p($8.98) → p = $0.67/$8.98 = 0.07461… = 7.461…% → 7.5%
    The sales tax rate is 7.5%.

17. a = pb → Discount = 0.40($45) = $18. Sale Price = Orig. Price - Discount $45 - $18 = $27
    The discount is $18, and the sale price is $27.

18. a = pb → Discount = 0.75($19.99) = $14.9925 → $14.99. Sale Price = $19.99 - $14.99 = $5.00
    Sales Tax = 0.055($5.00) = $0.275 → $0.28. Total Cost = $5.00 + $0.28 = $5.28
    The total cost of the CD is $5.28.

19. a = pb → Salary Cut = 0.15($52,000) = $7800. New Salary = $52,000 - $7800 = $44,200
    Eric's new annual salary will be $44,200.

20. a = pb → $805 = p($3500) → p = $805/$3500 = 0.23 = 23%
    23% of John's net salary goes toward housing expenses.

21. a = pb → $300 = 0.08(b) → b = $300/0.08 = $3750
    Jane's monthly net salary is $3750.

22. Groceries = 11% of $4100 → a = 0.11($4100) = $451
    Entertainment = 9% of $4100 → a = 0.09($4100) = $369

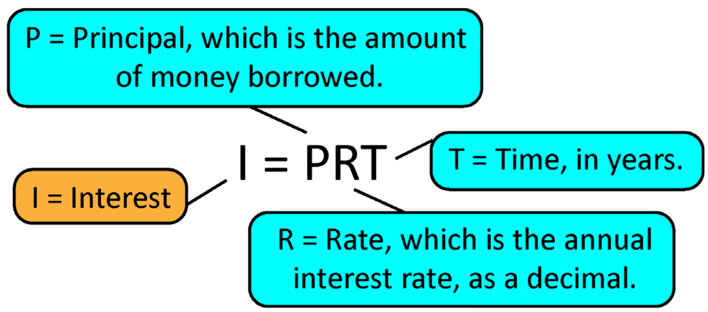
23. He spent 31% + 11% + 8% + 5% + 1% + 10% = 66%
    That means he saved 34% of his income.
    a = 0.34($3950) = $1343. Bob put $1343 into savings last month.

3.2: SIMPLE & COMPOUND INTEREST

Simple interest is just that: simple. It is calculated once during the loan period and is only computed upon the principal (money borrowed) of the loan or the investment. **Compound interest**, on the other hand, is calculated several times throughout the loan period. Furthermore, each time the interest is compounded, it is then added to the existing balance for the next compounding. That means we are actually paying (or earning) interest on interest!

Simple Interest

Interest is a percentage of a principal amount, calculated for a specified period, usually a stated number of years. Thus, the formula for simple interest is I = PRT.



- P stands for the **principal**, which is the original amount invested.
- R is the **annual interest rate**. Be sure to convert the rate to its decimal form for use in the formula.
- The **time**, T, is also referred to as the term of the loan or investment and is usually stated in terms of years.

It is possible to have the rate and time stated in terms of a time period other than years. If that is the case, make sure the two values are in agreement. That is, if we are working with a monthly rate, we need to state the time period in months, as well.

Also, remember, this formula gives us the amount of interest earned, so to find the total amount, A, of the loan or investment, we have to add that interest to the original principal, A = P + I.

Example 1: Find the total amount returned for a 5-year investment of $2000 with 8% simple interest.

First, find the interest. I = $2000(0.08)(5) = $800.
The total amount, A, returned to us is found by adding that interest to the original investment.
A = $2000 + $800 = $2800.

The amount returned at the end of the 5-year period will be $2800.

Pay attention to the question that gets asked. In some cases, we are seeking the total amount returned, A. In other cases, we may be asked to find just the interest. In the end, BE SURE to follow the instructions.

COMMON MISTAKE

One of the most common mistakes is neglecting to follow the instructions and answering the wrong question.

Example 2: Find amount of interest for a 5-year investment of $2000 with 8% simple interest.

I = $2000(0.08)(5) = $800.
The amount of interest on the investment will be $800.

Make sure the rate and time are both stated in years. If we are told an investment is for 18 months, we need to change that to 1.5 years for the computation.

Example 3: Joe borrowed $500 from his parents and agreed to repay it in six months with interest. The agreed upon APR was 2%. How much does Joe repay to his parents?

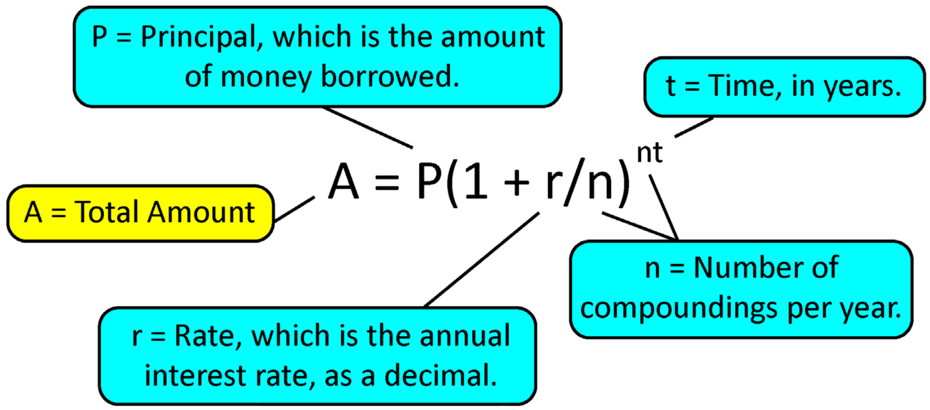
6 months = 0.5 years

I = $500(0.02)(0.5) = $5. That makes A = P + I = $500 + $5 = $505.
Joe will repay $505 to his parents.

Compound Interest

Legend has it, a Chinese Emperor was so enamored with the inventor of the game of chess, he offered the guy one wish. The inventor asked for something along the line of one grain of rice for the first square on the chessboard, two for the second square, four for the third, eight for the fourth, and so on - doubling for each of the 64 squares on the board. Thinking it was a modest request, the Emperor agreed. Upon finding out the inventor would receive $2^{64} - 1$ = 18,446,744,073,709,551,615 grains of rice (more than enough to cover the surface of the earth), he had the guy beheaded. Perhaps that story is what inspired **Albert Einstein** to refer to compound interest as the "most powerful force in the universe."

The formula for the compounded amount is $A = P(1+r/n)^{nt}$.



Don't fall into the trap of trying to memorize that formula. When we attempt to memorize something, we ultimately forget it or, even worse, recall it incorrectly. If we want to be able to recall a formula correctly, we need to understand the various components of it.

For the compounded amount formula, one of the most important things to realize is, unlike the formula for simple interest, the formula returns the total amount of the investment.

The compounded amount formula returns the entire compounded amount, A.
If we want to find only the interest, we need to subtract the original amount, I = A - P.

Just like in the simple interest formula, P stands for the principal, which is the original amount invested, and t stands for the time of the loan or investment and stated in terms of years.

In the compounded amount formula, the interest rate, r, is also referred to as the **Annual Percentage Rate** (**APR**). Be sure to convert the APR to its decimal form for use in the formula.

Be sure to change the APR to a decimal for use in the computation.

Next, pay special attention to n, the number of compoundings per year. In most instances, this is described by using terms like monthly, annually, or quarterly. Monthly means 12 times per year, quarterly is 4 times per year, semiannually is twice per year, and annually means once per year.

Compounded monthly means 12 times per year.
Compounded quarterly is 4 times per year.
Compounded semiannually is twice per year
Compounded annually means once per year.

Since n refers to the number of compoundings per year, r/n is the rate for each period. For example, if the APR is 15%, and the amount is compounded monthly, the monthly rate is 15%/12 = 1.25%. Do keep in mind, though; we still must use the APR in the formula.

We add 1 to the monthly rate to account for the interest being added to the original amount. This is similar to a sales tax computation. If an item sells for \$10, and the tax rate is 7%, the amount of the tax is \$10(0.07) = \$0.70, which would then be added to the original \$10 to get a total cost of \$10.70. Alternatively, we could compute it all in one step by multiplying \$10(1+0.07) = \$10(1.07) = \$10.70. The (1+r/n) in the compound amount formula is exactly like the (1+0.07) in that tax calculation we did earlier.

The exponent in the formula, nt, is the total number of compoundings throughout the life of the investment. For example, if the amount is compounded monthly for 5 years, there will be a total of 12(5) = 60 compoundings.

Example 4: Find the total amount due for \$2000 borrowed at 6% Annual Percentage Rate (APR) compounded quarterly for 3 years.

First, since quarterly implies 4 compoundings per year, r/n = 0.06/4 = 0.015. Then, add 1, and raise that sum to the power that corresponds to the number of compoundings. In this case, 4 times a year for 3 years is 12. Finally, multiply by the amount borrowed, which is the principal.

$A = \$2000\times(1 + 0.06/4)^{4\cdot 3} = \$2000\times(1.015)^{12} = \$2000\times1.195618 = \$2391.24.$

The total amount at the end of 3 years will be \$2391.24.

Using Your Calculator

In the previous example, we were faced with a multi-step calculation. Using a scientific calculator effectively can greatly expedite the process. For the exponentiation, $(1.015)^{12}$, we could literally multiply (1.015)(1.015)(1.015)…(1.015)(1.015) twelve times, but it is faster to use the exponentiation key on our calculator. Depending on the calculator, the exponentiation key may look like X^y or ^. Thus, $(1.015)^{12}$ would be done on a scientific calculator as 1.015 X^y 12, and then hit the = key. For more involved computations, we can use the parentheses keys. Ideally, we should never have to hit the = key more than once.

Example 5: Use a calculator to compute $\$1000(1 + 0.14/12)^{24}$.

Literally type: 1000 × (1 + 0.14 ÷ 12) X^y 24 =.

You should get 1320.9871…, which you will manually round to \$1320.99.

Example 6: On the day her son was born, Tricia invested \$2000 for him at a guaranteed APR of 8%, compounded monthly. How much will that investment be worth when he turns 50? How much will it be worth if he leaves it alone until he turns 65?

On his 50th birthday, the investment is worth:
$A = \$2000\times(1 + 0.08/12)^{12\cdot 50} = \$2000\times(1.0066666\ldots)^{600}$
$A = \$2000\times 53.878183179\ldots = \$107{,}756.37$

On his 65th birthday, the investment is worth:
$A = \$2000\times(1 + 0.08/12)^{12\cdot 65} = \$2000\times(1.0066666\ldots)^{780} = \$356{,}341.84$

Once again, we need to make sure to follow the instructions and answer the correct question. Sometimes we are asked to find the total amount, A, and other times we may be asked for just the interest, I.

Example 7: If \$1500 is invested at 5% APR compounded annually for 10 years, how much interest is earned?

The total amount, $A = \$1500\times(1 + 0.05/1)^{1\cdot 10} = \$2443.34$.

The interest earned is found by subtracting the value of the original investment:
$I = A - P = \$2443.34 - \$1500 = \$943.34$.

Section 3.2 Exercises

Unless directed otherwise, be sure to round money answers to the nearest cent and round percents to the nearest hundredth of a percent.

1. Find the simple interest on $12,000 for 2 years at a rate of 6% per year.

2. Find the simple interest on $25,000 for 6 months at a rate of 8.5% per year.

3. How much time is needed for $1800 to accumulate $360 in simple interest at a rate of 10%?

4. How much time is needed for $600 to accumulate $72 in simple interest at a rate of 4%?

5. If $4300 yields $1290 in simple interest over 6 years, what is the annual rate?

6. If $200 yields $45 in simple interest over 3 years, what is the annual rate?

7. How much principal is needed to accumulate $354.60 in simple interest at 9% for 4 years?

8. How much principal is needed to accumulate $65,625 in simple interest at 15% for 7 years?

9. If \$500 yields \$40 in simple interest over 2.5 years, what is the annual rate?

10. How much time is needed for \$1250 to accumulate \$375 in simple interest at a rate of 5%?

11. Find the simple interest on \$900 for 18 months at a rate of 9.5% per year.

12. If \$420 yields \$31 in simple interest over 30 months, what is the annual rate?

13. How much time is needed for \$660 to accumulate \$514.80 in simple interest at a rate of 12%?

14. Find the simple interest on \$1975 for 3 1/2 years at a rate of 7.2% per year.

15. Find the total amount, A, and the interest, I, for \$825 invested for 10 years at 4% APR, compounded annually.

16. Find the total amount, A, and the interest, I, for \$3250 invested for 5 years at 2% APR, compounded annually.

17. Find the total amount, A, and the interest, I, for $75 invested for 6 years at 3% APR, compounded semiannually.

18. Find the total amount, A, and the interest, I, for $1550 invested for 7 years at 5% APR, compounded semiannually.

19. Find the total amount, A, and the interest, I, for $625 invested for 12 years at 8% APR, compounded quarterly.

20. Find the total amount, A, and the interest, I, for $2575 invested for 2 years at 4% APR, compounded quarterly.

21. Find the total amount, A, and the interest, I, for $2575 invested for 2 years at 4% APR, compounded monthly.

22. Find the total amount, A, and the interest, I, for $1995 invested for 6 years at 5% APR, compounded semiannually.

23. Find the total amount, A, and the interest, I, for $460 invested for 7 years at 6% APR, compounded quarterly.

24. To purchase new equipment, Williams Brake Repair borrowed \$4500 at 9.5%. The company paid \$1282.50 in simple interest. What was the term of the loan?

25. John Doe earned \$216 simple interest on a savings account at 8% over 2 years. How much principal was initially in the account?

26. Michelle is 20 and wants to retire at the age of 50. If she invests a \$60,000 inheritance at 7% compounded monthly, how much will she have when she turns 50?

27. In order to pay for college, Brooke's parents invest \$20,000 in a bond that pays 8% interest compounded semiannually. How much money will there be in 18 years?

28. A couple sets aside \$5000 into a savings account that is compounded quarterly for 10 years at 9%. How much will the investment be worth at the end of 10 years?

Section 3.2 Exercise Solutions

1. I = PRT → I = ($12,000(0.06)(2) = $1440

2. I = PRT → I = ($25,000)(0.085)(0.5) = $1062.50

3. I = PRT → $360 = ($1800)(0.10)(T) → T = 360/180 = 2 years

4. I = PRT → $72 = ($600)(0.04)(T) → T = 72/24 = 3 years

5. I = PRT → $1290 = ($4300)(R)(6) → R = 1290/25800 = 0.05 = 5%

6. I = PRT → $45 = ($200)(R)(3) → R = 45/600 = 0.075 = 7.5%

7. I = PRT → $354.60 = (P)(0.09)(4) → P = $354.60/0.36 = $985

8. I = PRT → $65,625 = (P)(0.15)(7) → P = $65,625/1.05 = $62,500

9. I = PRT → $40 = ($500)(R)(2.5) → R = 40/1250 = 0.032 = 3.2%

10. I = PRT → $375 = ($1250)(0.05)(T) → T = 375/62.5 = 6 years

11. First note that 18 months is 1.5 years.
    I = PRT → I = ($900)(0.095)(1.5) = $128.25

12. 30 months = 2.5 years.
    I = PRT → $31 = ($420)(R)(2.5) → R = 31/1050 = 0.029523… = 2.9523…% → 2.95%

13. I = PRT → $514.80 = ($660)(0.12)(T) → T = 514.80/79.2 = 6.5 years (or 78 months)

14. I = PRT → I = ($1975)(0.072)(3.5) = $497.70

15. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded annually means n = 1.
    $A = \$825(1 + 0.04/1)^{10} = \$1221.2015…$ → $1221.20
    I = A – P = $1221.20 - $825 = $396.20

16. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded annually means n = 1.
    $A = \$3250(1 + 0.02/1)^{5} = \$3588.26261…$ → $3588.26
    I = A – P = $3588.26 - $3250 = $338.26

17. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded semiannually means n = 2.
    $A = \$75(1 + 0.03/2)^{12} = \$89.67136…$ → $89.67
    I = A – P = $89.67 - $75 = $14.67

18. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded semiannually means n = 2.
    $A = \$1550(1 + 0.05/2)^{14} = \$2190.109423…$ → $2190.11
    I = - A – P = $2190.11 - $1550 = $640.11

19. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded quarterly means n = 4.
    $A = \$625(1 + 0.08/4)^{48} = \$1616.91899… \rightarrow \$1616.92$
    $I = A – P = \$1616.92 - \$625 = \$991.92$

20. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded quarterly means n = 4.
    $A = \$2575(1 + 0.04/4)^{8} = \$2788.3560… \rightarrow \$2788.36$
    $I = A – P = \$2788.36 - \$2575 = \$213.36$

21. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded monthly means n = 12.
    $A = \$2575(1 + 0.04/12)^{24} = \$2789.09312… \rightarrow \$2789.09$
    $I = A – P = \$2789.09 - \$2575 = \$213.36$

22. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded semiannually means n = 2.
    $A = \$1995(1 + 0.05/2)^{12} = \$2683.05320… \rightarrow \$2683.05$
    $I = A – P = \$2683.05 - \$1995 = \$688.05$

23. Since it involves compound interest, we use $A = P(1 + r/n)^{nt}$. Compounded quarterly means n = 4.
    $A = \$460(1 + 0.06/4)^{28} = \$697.9222… \rightarrow \$697.92$
    $I = A – P = \$697.22 - \$460 = \$237.92$

24. “Simple interest” is stated, so use I = PRT.
    $I = PRT \rightarrow \$1282.50 = (\$4500)(0.095)(T) \rightarrow T = 1282.50/427.50 = 3$ years

25. “Simple interest” is stated, so use I = PRT.
    $I = PRT \rightarrow \$216 = (P)(0.08)(2) \rightarrow P = \$216/0.16 = \$1350$

26. “Compounded monthly” is stated, so use $A = P(1 + r/n)^{nt}$ with n = 12. Also, T = 30 years.
    $A = P(1 + r/n)^{nt} \rightarrow A = \$60{,}000(1 + 0.07/12)^{360} = \$486{,}989.8479… \rightarrow \$486{,}989.85$

27. “Compounded semiannually” is stated, so use $A = P(1 + r/n)^{nt}$ with n = 2. Also, T = 18 years.
    $A = P(1 + r/n)^{nt} \rightarrow A = \$20{,}000(1 + 0.08/2)^{36} = \$82{,}078.65108… \rightarrow \$82{,}078.65$

28. “Compounded quarterly” is stated, so use $A = P(1 + r/n)^{nt}$ with n = 4. Also, T = 10 years.
    $A = P(1 + r/n)^{nt} \rightarrow A = \$5000(1 + 0.09/4)^{40} = \$12{,}175.9448… \rightarrow \$12{,}175.94$

3.3: INSTALLMENT BUYING & CREDIT CARDS

Installment Buying

Installment buying is the process of purchasing something and paying for it over a period of time. Do not confuse this with **Lay-Away buying**, in which a customer makes monthly payments and then obtains the product after it is paid for in full.

For the convenience of paying later, the consumer normally has to pay a finance charge. Also, the terms of any such agreement must, by law, be disclosed in writing at the beginning of the process. More information about the **Truth-in-Lending Act (TILA)** can be found online in the **Wikipedia** entry at http://en.wikipedia.org/wiki/Truth_in_Lending_Act.

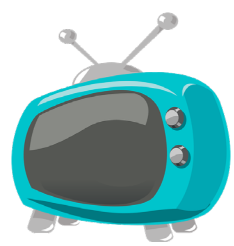
To simplify the calculations in this section, we will not consider any applicable taxes that would normally be imposed on the indicated purchases. Conceptually, the calculations are the same with or without those taxes. We will also assume the regular payments do not contain any extra funds.

Simple Finance Charges

Normally, borrowing money or having the privilege of making regular payments is not free. The **finance charge** is the amount of the purchase in excess of the selling price. Sometimes the finance charge is a straight fee, and other times it is a percentage of the purchase price. In all cases, however, it is the amount in excess of the selling price.

The finance charge imposed for the privilege of making regular payments is the amount paid in excess of the purchase price.

Example 1: Joel wants to purchase a TV for $1075. Since he does not have the cash up front, he enters into an agreement to pay $35 a month for 36 months. Find the total cost and the finance charge for this purchase.



The total amount he will pay is ($35)(36) = $1260.
Then, finance charge would be $1260 - $1075 = $185.

Calculating Monthly Payments

A critical component of installment buying is the calculation of the **monthly payment**. The tricky part of determining a monthly payment is realizing the last payment will usually be slightly more or slightly less than all the other monthly payments.

Example 2: A recliner is priced at $450, but it may be purchased on an installment plan by paying $100 down, and then paying the balance plus 18% simple interest in 12 monthly payments. What would be the amount of each payment?

Using the simple interest formula, we will calculate the amount of interest that is to be paid. Remember, the interest rate has to be in decimal form and the time is stated in years.
I = PRT → I = ($350)(0.18)(1) = $63.

Adding that $63 to the balance of $350, the total amount of the payments will be $413. Since there will be 12 payments, each one should be $413/12 = $34.416666… → $34.42

Note: The first 11 payments would be $34.42, and the last payment would be the remaining amount of the purchase. 11×$34.42 = $378.62, and $413 - $378.62 = $34.38.
The last payment on the purchase will be $34.38.

Example 3: Sarah buys a new mountain bike for $375 and pays for it over two years with 13.5% simple interest. What is her monthly payment?

To find the amount of interest she will pay, use the simple interest formula: I = ($375)(0.135)(2) = $101.25

This brings the total cost for the bike up to $476.25. Since she will be making 24 payments, $476.25/24 = $19.84375. That means the first 23 payments will be $19.84, and the last one will be $19.93.

In the two previous examples, the assumption is that no additional money is paid on top of the monthly payment. That is, if the payment happens to be $19.84, we are assuming the amount paid is exactly $19.84, and not a convenient $20.

Credit Card Finance Charges

If we pay off a credit card balance each month, we are not charged any finance charges. If, however, we carry a balance to the next month, the finance charge on a **credit card** is the simple interest on the average daily balance using a daily interest rate. In such a calculation, we also need to know how many days are in the billing cycle.

The **billing cycle** is the number of days between credit card statements, and usually corresponds to the number of days in a specific month, which will by 28 (29 in a leap year), 30 or 31 days. We could pull out a calendar and count the days, but a faster way is to simply recognize the month that we are considering, and look to see if there are 30 or 31 days in it.

Example 4: How many days are in the billing cycle that runs from June 15 through July 14?

This billing cycle crosses over the end of June. Since there are 30 days in June, there are 30 days in billing cycle.

Mon	Tue	Wed	Thr	Fri	Sat	Sun
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30		

The **average daily balance (ADB)** is just that - the average of the daily balances for all the days in the billing cycle. We could find the ADB by finding the balance for every day in the billing cycle, adding them all together, and then dividing that sum by the number of days in the billing cycle. If we see the balance remain unchanged for stretches of several days at a time, the calculation can be done more efficiently.

Example 5: If a credit card has a balance of $20 for 21 days and then a balance of $40 for 9 days, what is the average daily balance for that 30-day period?

Rather than adding together $20 + $20 + $20 + $20 + … + $20 + $20 (21 times), we can simply multiply 21 × $20 to get $420. Likewise, for the 9-day stretch, the sum of those daily balances will be 9 × $40 = $360. For the entire 30-day period, the sum of all 30 individual balances will be $780. Then, the ADB for that 30-day period will be $780/30, which is $26.

Example 6: A credit card has a balance of $200 for 11 days, then a balance of $100 for 6 days, and then a balance of $150 for 15 days, what is the average daily balance for that 31-day period?

ADB = (11×$200 + 6×$100 + 15×$150)/31 = $5050/31 = $162.90322… → $162.90

The next piece of the credit card interest puzzle is the **daily percentage rate (DPR)**. The interest rate stated with a credit card is always an annual percentage rate (APR). Since there are 365 days in a year (366 in a leap year), the daily interest rate is the APR/365.

Putting it all together, the monthly finance charge, FC, on a credit card is the product of the Average Daily Balance, the Daily Percentage Rate, and the number of days in the cycle.

Monthly Finance Charge = (ADB)×(DPR)×(# of Days in Billing Cycle)

Example 7: Let's say we have an average daily balance of \$183.65 on a credit card with an annual percentage rate (APR) of 16.9%, over a billing period of 31 days. What would the finance charge be?

Remember, we need to change the APR to decimal.

FC = ADB × DPR × # of Days = \$183.65 × (0.169/365) × 31 = \$2,63600… → \$2.64

Using Your Calculator

You may have picked up on that fact that many of the calculations in this section have a lot of steps. As with any multi-step calculation, there are a lot of opportunities for simple mistakes, and it only takes one simple mistake to throw off the final answer. One of the most common simple mistakes is rounding at an intermediate step.

COMMON MISTAKE

A very common mistake is rounding at an intermediate step.

Our answers will be the most accurate if we only round once, and at the very end of the problem.

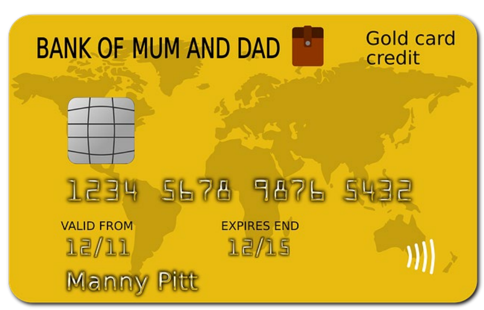
One way to combat this is to make effective use of our calculator. In the previous example, we computed daily percentage rate as 0.169/365. If we did that calculation by itself as 0.000463013…, and used a rounded version, such as 0.00046, in our calculation for the next step, we would have got an answer of \$2.62 instead of \$2.64. Sure, the difference is just a couple pennies, but it is still incorrect.

Instead of doing the calculation piecemeal, we should take full advantage of the features of the scientific calculator and do the entire calculation in one long computation, using the maximum number of digits at each intermediate step, and hitting the equals key only once. Then, since the answer is money, we round to the nearest cent at the very end.

Credit Card Balance Calculations

Finally, once we can correctly identify the number of days in a billing cycle, find daily percentage rates, and accurately compute monthly finance charges on unpaid balances, we are ready for the last step: we need to add the finance charges to the existing balance.

Don't accidentally add it to the Average Daily Balance.



COMMON MISTAKE

A common mistake is to add the finance charge to the ADB instead of the existing balance.

Here are a couple of examples.

Example 8: The balance on the Doe's credit card on May 12, their billing date, was \$378.50. They made a \$50 payment that was recorded on May 15, and then bought \$37.63 in gas on June 1. The APR for the credit card is 11.5%. Find the balance due on June 12.

The very first thing we need to note is the number of days in the billing cycle. Since the cycle crosses over the end of May, there are 31 days in the cycle.

Next, we need to note the balance on certain stretches in the cycle.

- 5/12 through 5/14: 3 days with a balance of \$378.50
- 5/15 through 5/31: 17 days with a balance of \$328.50
- 6/1 through 6/11: 11 days with a balance of \$366.13

ADB = (3×\$378.50 + 17×\$328.50 + 11×\$366.13)/31 = \$10,747.43/31 = \$346.69

FC = ADB × DPR × # of Days = \$346.69 × (0.115/365) × 31 = \$3.39

Balance Due = \$366.13 + \$3.39 = \$369.52

Example 9: The balance on the Smith's credit card on November 8, their billing date, was \$812.96. They charged \$231.10 in toys on November 13, and made a \$500 payment that was recorded on November 15. If the APR for the card is 5.9%, what is the balance due on December 8?

Since the cycle crosses over the end of November, there are 30 days in the cycle.

Next, we need to note the balance on certain stretches in the cycle.

- 11/8 through 11/12: 5 days with a balance of \$812.96
- 11/13 through 11/14: 2 days with a balance of \$1044.06
- 11/15 through 12/7: 23 days with a balance of \$544.06

ADB = (5×\$812.96 + 2×\$1044.06 + 23×\$544.06)/30 = \$18,666.30/30 = \$622.21

FC = ADB × DPR × # of Days = \$622.21 × (0.059/365) × 30 = \$3.02

Balance Due = \$544.06 + \$3.02 = \$547.08

Credit Cards in General

Credit cards can be very useful. Unfortunately, they are also very dangerous. In the late 1980s credit card companies would target unsuspecting college students who were just a year or two away from their parents' watchful eyes. Many times, those students did not even have jobs or a regular income; the companies were issuing the credit cards to the students based solely on potential income. In turn, many college students would rack up thousands of dollars of debt without even realizing the consequences. New credit card laws that went in to affect in February of 2010 specifically attempt to limit this practice with a few restrictions. Credit card companies are now banned from issuing cards to anyone under 21, unless they have adult co-signers on the accounts or can show proof they have enough income to repay the card debt. The companies must also stay at least 1,000 feet from college campuses if they are offering free pizza or other gifts to entice students to apply for credit cards.

Other highlights from the **Credit CARD Act** (CARD stands for Card Accountability Responsibility and Disclosure) include limits on interest rate hikes and fees, allowing card holders at least 21 days to pay monthly bills, mandates on due dates and times, and clear information on the consequences of making only minimum payments each month. For that last one, the credit card companies must identify how long it would take to pay off the entire balance if users only made the minimum monthly payment.

If you are just starting out on your own, it is a good idea to avoid the overuse of credit cards. If you do not have the money, give a second thought to the purchase. Is it something you really need? Instead, try to think of a credit card as a cash substitute. If you have the money to cover the purchase, the credit card can allow you to leave the money in the bank and not carry it with you. After you make the purchase, you can pay off the balance on the credit card immediately after the purchase. Furthermore, if you pay the entire balance on a credit card each month, most credit card companies do not impose any finance charges. Above all, don't spend money you do not have.

Section 3.3 Exercises

Be sure to round all money answer to the nearest cent. Be aware that some of the following exercises have multiple parts. Be sure to answer all the questions that are asked.

1. Mary bought a used car for $3500, which was financed for $160 a month for 24 months. What is the total cost for the car? How much is the finance charge?

2. Jose borrowed $800 from his local credit union for 8 months at 6% simple annual interest. He agreed to repay the loan by making eight equal monthly payments. How much is the finance charge? What is the total amount to be repaid? How much is the monthly payment?

3. Pam purchased a new laptop, which was advertised for $1200. She bought it on the installment plan by paying $200 at the time of purchase and agreeing to pay the rest plus 13% simple annual interest on the balance in 24 monthly payments. How much is the finance charge? How much will each payment be? What is the total cost of the laptop?

4. Joe buys a new stereo for $875 and pays for it over two years with 6.5% simple annual interest. What is his monthly payment?

5. How many days are in a credit card billing cycle that runs from July 17 through August 16?

6. How many days are in a credit card billing cycle that ends on December 12?

7. How many days are in a credit card billing cycle that begins on august 22?

8. If a credit card has a balance of $30 for 11 days and then a balance of $50 for 20 days, what is the average daily balance for that 31-day period?

9. If a credit card has a balance of $120 for 10 days, a balance of $150 for the next 12 days, and a balance of $90 for 8 days, what is the average daily balance for that 30-day period?

10. If the APR on a credit card is 6.57%, what is the daily interest rate? Round your answer to the nearest thousandth of a percent.

11. If the APR on a credit card is 8.75%, what is the daily interest rate? Round your answer to the nearest thousandth of a percent.

12. The average daily balance is $210.39 on a credit card with an APR of 13.4%, over a billing period of 30 days. What will the finance charge be?

13. The average daily balance is $1210.34 on a credit card with an APR of 6.4%, over a billing period of 31 days. What will the finance charge be?

14. The average daily balance is $90.15 on a credit card with an APR of 5.4%, over a billing period of 30 days. What will the finance charge be?

15. The balance on Maria's credit card on May 10, the billing date, was $ 3198.23. She sent in a $1000 payment, which was posted on May 13, and made no other transactions during the billing cycle. Assuming the APR on the card is 7.9%, what will be the new balance when she receives her June 10 statement?

16. The balance on Jim's credit card on April 17, the billing date, was $ 78.30. He sent in a $50 payment that was posted on April 20, and then charged $29.20 in gas on May 1. Assuming the APR on the card is 5.2%, what will be the balance at the beginning of the next cycle?

17. The balance on Carrie's credit card on November 19, the billing date, was \$238.50. She sent in a \$150 payment that was recorded on November 23, and then made a \$129.19 charge on December 1. Assuming the APR on the card is 2.9%, what will be the balance at the beginning of the next cycle?

18. The balance on Ted's credit card on August 9, the billing date, was \$1298.51. He sent in a \$350 payment that was recorded on August 14, and then charged \$87.17 in groceries on August 29. Assuming the APR on the card is 3.9%, what will be the balance at the beginning of the next cycle?

19. The balance on Xavier's credit card on August 19, the billing date, was \$300. He sent in a \$100 payment that was recorded on August 26 and didn't have any other transactions in the cycle. Assuming the APR on the card is 9.9%, what will be the balance at the beginning of the next cycle?

Section 3.3 Exercise Solutions

1. Total Cost = 24($160) = $3840
   FC = $3840 - $3500 = $340

2. Use T = 8/12 Years
   FC = $800(0.06)(8/12) = $32
   Total Amount = $800 + $32 = $832
   Monthly Payment = $832/24 = $104

3. Use T = 2 Years and note that $1000 was financed.
   FC = $1000(0.13)(2) = $260
   Monthly Payment = ($1000 + $260)/24 = $52.50
   Total Cost = $200 + $1000 + $260 = $1460

4. Monthly Payment = [$875 + $875(0.065)(2)]/24 = $41.1979… → $41.20
   (Note: The last monthly payment will be $41.15.)

5. The cycle crosses over the end of July, which means it has 31 days.

6. The cycle crosses over the end of November, which means it has 30 days.

7. The cycle crosses over the end of August, which means it has 31 days.

8. ADB = (11×$30 + 20×$50)/31 = $42.90322… → $42.90

9. ADB = (10×$120 +12×$150 + 8×$90)/30 = $124.00

10. DPR = APR/365 = 6.57%/365 = 0.018%

11. DPR = APR/365 = 8.75%/365 = 0.02397…% → 0.024%

12. FC = ADB × DPR × # of Days = $210.39 × 0.134/365 × 30 = $2.317172… → $2.32

13. FC = ADB × DPR × # of Days = $1210.34 × 0.064/365 × 31 = $6.57894… → $6.58

14. FC = ADB × DPR × # of Days = $90.15 × 0.054/365 × 30 = $0.40011… → $0.40

15. The cycle crosses over the end of May, so it has 31 days.
    - 5/10-5/12: 3 days at $3198.23
    - 5/13-6/4: 28 days at $2198.23
    ADB = (3×$3198.23 + 28×$2198.23)/31 = $2295.00419… → $2295.00
    FC = ADB × DPR × # of Days = $2295.00 × 0.079/365 × 31 = $15.39850… → $15.40
    BAL FWD = $2198.23 + $15.40 = $2213.63

16. The cycle crosses over the end of April, so it has 30 days.
    - 4/17-4/19: 3 days at $78.30
    - 4/20-4/30: 11 days at $28.30
    - 5/1-5/16: 16 days at $57.50

    ADB = (3×$78.30 + 11×$28.30 + 16×$57.50)/30 = $48.87333... → $48.87
    FC = ADB × DPR × # of Days = $48.87 × 0.052/365 × 30 = $0.208869... → $0.21
    BAL FWD = $57.50 + $0.21 = $57.71

17. The cycle crosses over the end of November, so it has 30 days.
    - 11/19-11/22: 4 days at $238.50
    - 11/23-11/30: 8 days at $88.50
    - 12/1-12/18: 18 days at $217.69

    ADB = (4×$238.50 + 8×$88.50 + 18×$217.69)/30 = $186.014... → $186.01
    FC = ADB × DPR × # of Days = $186.01 × 0.029/365 × 30 = $0.4433... → $0.44
    BAL FWD = $217.69 + $0.44 = $21813

18. The cycle crosses over the end of August, so it has 31 days.
    - 8/9-8/13: 5 days at $1298.51
    - 8/14-8/28: 15 days at $948.51
    - 8/29-9/8: 11 days at $1035.68

    ADB = (5×$1298.51 + 15×$948.51 + 11×$1035.68)/31 = $1035.8929... → $1035.89
    FC = ADB × DPR × # of Days = $1035.89 × 0.039/365 × 31 = $3.4312... → $3.43
    BAL FWD = $1035.68 + $3.43 = $1039.1113

19. The cycle crosses over the end of August, so it has 31 days.
    - 8/19-8/25: 7 days at $300
    - 8/26-9/18: 24 days at $200

    ADB = (7×$300 + 24×$200)/31 = $222.5806 → $222.58
    FC = ADB × DPR × # of Days = $222.58 × 0.099/365 × 31 = $1.8715... → $1.87
    BAL FWD = $222.58 + $1.87 = $201.8713

3.4: LEASES

General Lease Information

A **lease** is a contract that allows for one party to use the property of another party, for an extended period of time. Leases can be written for just about anything, but generally include cars, heavy equipment, general office equipment, housing or office space.

The **lessee** is the party or individual wanting to utilize the property, and the **lessor** is owner of the property. A lease involving tangible property, such as a car or real estate, is also referred to as a **rental agreement**, and leases involving non-tangible property, such as cell phone airtime, are usually referred to as **contracts**. Whether a rental agreement or a contract, the lessee will pay the lessor a monthly fee for the property's usage. A lease indicates legal conditions and responsibilities of both the lessee and lessor.

Rental Agreements

Probably the most common type of lease is a real estate rental agreement for residential or business use, with the lessor being referred to as a **landlord** and lessee being called the **tenant**. **Rent** payments on the property are usually made on a monthly basis.

In a **fixed-term tenancy**, either the landlord or the tenant may terminate a rental agreement when the specified term is nearing completion, but neither party may terminate the lease early without a penalty based on the remaining period of the lease.

An **at-will tenancy** is a tenancy in which either the landlord or the tenant may terminate at any time by giving reasonable notice. Either the lessor or the lessee for any reason, or for no reason can end it at any time at all.

The **security deposit** is an initial payment paid by the tenant and held by the landlord until the property is returned or vacated. Normal wear and tear is expected and a list of pre-existing damage to the property is provided before the tenancy begins. Upon the termination of the rental agreement, if the property is returned in good condition, all or part of the security deposit is returned to the tenant. Depending on the language in the rental agreement, charges for cleaning or other fees may be deducted before the deposit is returned.

Example 1: A property owner is asking for a \$1000 security deposit and \$850/month rent. What will be the total cost for a two-year lease?

Total Cost = \$1000 + 24(\$850) = \$21,400

Rentable **office spaces** are primarily classified as Class A, Class B or Class C. **Class A** office spaces are found in the best quality buildings and the most desirable locations. They are in newer, well-constructed, large buildings with convenient access, and are professionally managed. Thus, they are the most expensive to rent.

Class B office spaces are found in buildings are well-maintained, but also a little older. They still attract good tenants and provided at more affordable rents than their Class A counterparts. Some Class B buildings were originally Class A facilities and, hence, with renovations, they can be returned to Class A status.

Older buildings in need of extensive repairs and are found in less desirable areas are **Class C** office spaces. Although offering the lowest rental rates, these buildings often need expensive infrastructure and technology upgrades to be suitable for many newer companies.

Example 2: A company owns a Class B building and charges $3000 per month to rent it. If the company spends $60,000 to upgrade the facility, by how much would they have to increase the monthly rent to recover the cost of the upgrade within 5 years?

5 years = 60 months
$60,000/60 months = $1000/month
The new rent in the upgraded facility would need to be $4000/month.

The Class of the building is just one factor in the rental process. In addition to the general age and size of the usable space, we also need to consider issues like the capacity of the wiring to handle computer networks, and even security. Different companies have different needs, so careful consideration must be given to the prioritization of the facilities amenities. And, above all, be sure to have the property inspected by a professional.

Writing a Lease Agreement

Lease agreements are intended to protect both the landlord and the tenant from unexpected problems. These agreements do not have to be complicated. A succinct and easy-to-understand lease can often be much better than a long and overly complicated one. If we attempt to write a document that addresses many specific items, it will be very easy to overlook a number of things. By keeping the agreement in general terms, we can avoid oversights. For example, if we do not want a tenant to keep a pet in the rental unit and we write into the lease, "No dogs, cats, reptiles, goats, rodents, fish or birds," we have unintentionally allowed the tenant to keep a pot-bellied pig as a pet. If, however, we simply write, "No pets allowed." into our agreement, that should not be a problem. In other words, be thorough but tried to avoid specifics.

Disputes over the return of the security deposit can easily become very contentious. If we are writing a lease, and we clearly indicate costs to perform property repairs (beyond normal wear and tear) and cleaning charges that will be deducted from the security deposit, we can avoid many problems. Above all, make sure the tenant fully understands the included language to this regard.

Finally, it would be very wise to have our lease reviewed by a real-estate lawyer to ensure the agreement is legally binding and complies with all state laws. A relatively small fee paid to an attorney up front will be much more affordable than dealing with a poorly written lease a few years down the road.

Property Management Companies

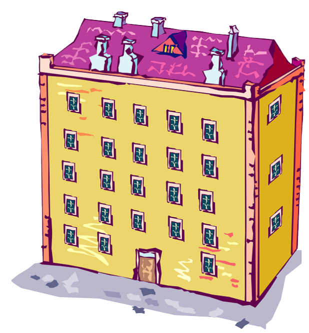
If we are venturing into the property rental arena for the first time, a **property management company** may prove to be extremely beneficial. Depending on the amount of service they provide – ranging from tenant screening, lawn care maintenance, and fiscal management – property management companies usually charge between 5-10% of the monthly rent. Some even charge a fixed fee, even if the property is not rented. If the property owners live in a different area, having a company manage the property for them can be a nice convenience.

Example 3: Mr. & Mrs. Smith have moved to New York and wish to rent their home in Dallas, so they hire a management company that charges 8% of the rental price for their fee. If the home rents for $1200/month, how much will Smiths earn from the monthly rent?

The company charges 8% of $1200, which is $96. The rest, $1104, is sent to the Smiths.

Sometimes, apartment buildings or complexes have managers that live on site. Those individuals are usually paid by keeping a percentage of the rent collected from the tenants.

Example 4: Howard manages an apartment building with 16 apartments in it. The rent on each apartment is $800/month, of which Howard keeps 28%. How much does Howard make in a year?



Form an individual apartment, Howard makes 28% of 800 each month. From there, multiply by 16 apartments and, since we are looking for the annual income, multiply by 12.

Annual Income = (0.28)($800)(16)(12) = $43,008

Leasing a Car or Equipment

Leasing cars or equipment is an attractive option for many people or companies. Due to depreciation and wear and tear, some large items lose a significant portion of their value over extended periods. Furthermore, as the ages of these things increase, annual maintenance costs escalate, as well. Before we blindly dive into a lease, we should, however, carefully consider four basic things:

1. The Up-Front Payment
2. The Monthly Payment
3. The Length of the Lease
4. Possible Additional Charges at the End of the Lease

Example 5: Mike leases a new pickup by paying $3000 up front and $249 a month over three years. The lease also stipulates he will be charged $0.15 per mile for every mile over 36,000. If he puts 40,196 miles on the truck, what will be the total cost of the lease?

Noting the excess mileage is 40,196 – 36,000 = 4196 miles,
Total Cost = $3000 + 36($249) + 4196($0.15) = $12,593.40

Example 6: Deanna wants to lease a new Toyota Camry, and she is presented with the following three options. Compute the total cost of each option and determine which one is best for her. Provided the vehicle has less than 30,000 miles on it by the end of the lease, there will not be any additional charges.

- Option 1: $0 up-front, with a payment of $209/month for 36 months.
- Option 2: $999 up-front, with a payment of $189/month for 36 months.
- Option 3: $2999 up-front, with a payment of $149/month for 36 months.

- Option 1: $0 + $209(36) = $7524
- Option 2: $999 + $189(36) = $7803
- Option 3: $2999 + $149(36) = $8363

With all other things being equal, the option Deanna chooses will be based on her financial situation and cash reserves at the time of the lease. If she really wants the smallest monthly payment and has the $2999 to pay up-front, Option 3 may be her best choice. If, however, she can afford the larger monthly payment, she would save over $800 by choosing Option 1.

If we wish to keep a newer car or piece of equipment, we should compare the cost associated with leasing, and the costs associated with buying - and then selling - the same item over the same period.

Example 7: The new Camry Deanna is considering leasing can be purchased for a down payment of $4000 and 36 monthly payments of $550. After 3 years, she will own the car and, based on projections, she will be able to sell it for $16,000. If she follows this plan, what will be the total net cost for the car?

The total net cost will be the amount paid less the amount recovered.

The amount paid will be $4000 + $550(36) = $23,800.
Thus, $23,800 - $16,000 = $7800.

Taking another look at the previous examples, Lease Option 1 offers a lower total net cost, no up-front or down payment costs, monthly payments that are less than half of the purchase payments, and no re-sale headaches at the end of three years. Thus, if Deanna wants to get a new car in three years, leasing is the much better option. If, however, she wishes to keep her Camry for 5-6 years, having it paid off in three years means there will no longer have a monthly car payment expense.

Be aware, since we would pay tax, title, license and other documentation fees whether we leased or bought a car, they are not factored into our calculations. We would also need to check with our insurance company to see if they offered different rates for leasing and buying the same vehicle.

When taking out a loan to purchase a car, the term of the loan is typically 5 years (60 months). The Camry in Example 5 could be paid for over 60 months with a smaller payment (around $350), but a longer term also means more interest paid on the load and, hence, a higher total net cost. The additional costs associated with a 60-month loan do not mean they are not worth it. In fact, if the vehicle maintains a higher resale value, the average annual cost associated with the longer-term plan may actually make it a better option.

Be very careful with leases. Even though the allure and peace of mind that comes with driving a new car every few years may make the extra expenses worth considering. We do, however need to realize the company leasing the vehicle is looking to make money. Many leases may be advertised as having “no down payment,” but may include other up-front costs. Also, be prepared to pay substantial penalties if a specified mileage is exceeded, or another term violated in the agreement. Due to complex and clever language, it is not uncommon for the total cost of a lease to be much, much more than anticipated.

Some leases also give us the option of buying the vehicle at the end of the contract. Thus, if we really like the car, we may be able to keep it. Unfortunately, this usually leads to a higher total cost than buying it outright. According to LeaseTrader.com, if we are planning on keeping the car for more than 4 years buying it outright might be the better choice. However, if we do not want to keep the car for more than 4 years we should give serious and careful consideration to a lease.

Leasing equipment – everything from cranes to computers - is an attractive option for many businesses. If a piece of equipment is purchased, it may be very difficult to upgrade or sell if it no longer meets the needs of the business or becomes outdated. Smaller payments and the ability to adapt to changing circumstances help businesses stay competitive.

Section 3.4 Exercises

1. A property owner is asking for a \$1500 security deposit and \$950/month rent. What will be the total cost for a two-year lease?

2. A property owner is asking for a \$700 security deposit and \$550/month rent. What will be the total cost for a three-year lease?

3. Joe estimates he can afford a total of \$9000 a year in rent. What can he afford as a monthly payment?

4. Sondra estimates she can afford a total of \$15,000 a year in rent. What can she afford as a monthly payment?

5. A company rents an office building for \$2500 per month. If they spend \$60,000 to upgrade the facility, by how much would they have to increase the monthly rent to recover the cost of the upgrade within 10 years? How much would they increase the monthly rent to recover the cost of the upgrade in 5 years?

6. A company owns a Class B building and charges \$10,000 per month to rent it. If the company spends \$120,000 to upgrade the facility to a Class A building, by how much would they have to increase the monthly rent to recover the cost of the upgrade within 5 years?

7. Ms. Landis has a condo she plans to rent out. She hires a management company that charges 9% of the rental price for their fee. If the condo rents for \$1500/month, how much will Ms. Landis earn from the monthly rent?

8. Jennifer has a home she plans to rent out. She hires a management company that charges 10.5% of the rental price for their fee. If the home rents for $1800/month, how much will she earn from the monthly rent?

9. Quality Management charges 8.5% of the monthly rent on properties it manages. If they manage nine different condos that each rent for $1300/month, what is the total annual amount they earn from those properties?

10. A new Hyundai Sonata can be leased for $199/month for 36 months, with $2,399 due up front. Assuming no additional charges will occur, what will be the total cost of this lease?

11. A new Ford Escape can be leased for $274/month for 39 months, with $3076 due up front. Assuming no additional charges will occur, what will be the total cost of this lease?

12. A new Honda Accord can be purchased for a down payment of $2000 and 60 monthly payments of $350. What will be the total cost for this car? What will be the average cost per year?

13. A new Cadillac CTS Coupe can be leased for an up-front payment of $2999 and 36 monthly payments of $389. What will be the total cost for this car? What will be the average cost per year?

14. Dan leased a Ford truck by paying $3500 up front and, for the last three years, he has made monthly payments of $189. The lease stipulated he would be charged $0.20/mile for any mileage in excess of 36,000. When he returned the truck, it had 39,851 miles on it. What was the total cost of the lease?

15. To avoid a large purchase, a start-up company decides to lease 8 notebook computers for its salesmen. They pay $135/month for each machine for 24 months. What will be the total cost of the computers?

16. Home Depot charges $19.95 to rent a truck for 75 minutes, with an additional charge of $5 for every 15 minutes, thereafter. How much will it cost to rent the truck for 3 hours?

17. U-Haul charges $19.95 a day plus $0.69/per mile to rent a 10-ft truck, and $29.95 plus $0.69/mile to rent a 14-ft truck. Jim's new house is 8 miles away from his old house, and the truck rental company is exactly halfway between the two houses. He estimates he can make the move in two trips with the smaller truck and one trip with the bigger truck. Find the total cost for each truck rental and indicate which one will cost less money.

Answers to Section 3.4 Exercises

1. Total Cost = $1500 + 24($950) = $24,300

2. Total Cost = $700 + 36($550) = $20,500

3. Maximum Monthly Rent = $9000/12 = $750

4. Maximum Monthly Rent = $15,000/12 = $1250

5. $60,000/120 months means they need to collect an extra $500/month for the 10-yr recovery
   For the 5-yr recovery, they need to collect an extra $60,000/60 months = $1000/month.

6. For a 5-yr recovery, they need to collect an extra $120,000/60 = $2000/month.

7. The management company keeps 9%, which means Ms. Landis gets 91%.
   (0.91)($1500) = $1365

8. The management company keeps 10.5%, which means Jennifer gets 89.5%.
   (0.895)($1800) = $1611

9. (0.085)($1300)(9)(12) = $11,934

10. $2399 + 36($199) = $9563

11. $3076 + 39($274) = $13,762

12. Total Cost = $2000 + 60($350) = $23,000. $23,000/5 yr = $4600/year

13. Total Cost = $2999 + 36($389) = $17,003. $17,003/3 yr = $5667.67/year

14. The excess mileage is 39,851 – 36,000 = 3851.
    Total Cost = $3500 + 36($189) + 3851($0.20) = $11,074.20

15. Total Cost = 8($135)(24) = $25,920

16. 75 minutes is 1 hr 15 min. So, for a 3-hr rental, the first 1 hr 15 min is $19.95. The remaining 1 hr 45 min has to be broken into 15-min intervals, of which there are 7.
    Total Cost = $19.95 + 7($5) = $54.95

17. For the smaller, 10-ft truck, he picks up the truck and drives 4 miles to the old house. After loading it, he drives 8 miles to the new house. Then he returns to the old house, loads it, and then drives to the new house again. After unloading it for the second time, he drives 4 miles to return the truck. That’s 32 miles. The rental fee is then $19.95 + 32($0.69) = $42.03.

    For the larger, 14-ft truck, eliminate the second trip. That gives us 16 miles.
    The rental fee is $29.95 + 16($0.69) = $40.99. The 14-ft truck costs less.

3.5: MORTGAGES

'Til Death Do Us Part

In French, the word *mort* means death and the word *gage* means pledge. Thus, the literal meaning of the word **mortgage** is "death pledge." Now that's something to think about…

Amortization

Amortization is a situation in which the borrower agrees to make regular payments on the principal and interest until a loan is paid off. There are a lot of variables involved in obtaining a home loan, and each should be weighed carefully before entering into such a major commitment.

Amortization Tables

Somewhere, someone once sat down and calculated the monthly payments for loans using different amounts, interest rates, and time periods. Extensive **amortization tables** are rarely seen, but they are pretty large and fairly organized. Succinct versions of these tables are a nice way to quickly compare calculations for different time periods. If you use a table computed on a fixed amount of $1000, you can scale the amount to match any principal, being sure to round the calculated amount to the nearest cent.

Here is a partial amortization table.

Table 1: Monthly Mortgage Payment per $1000

Rate (%)	15 Years	20 Years	30 Years
4.50	7.6499	6.3265	5.0669
4.75	7.7783	6.4622	5.2165
5.00	7.9079	6.5996	5.3682
5.25	8.0388	6.7384	5.5220
5.50	8.1708	6.8789	5.6779
5.75	8.3041	7.0208	5.8358
6.00	8.4386	7.1643	5.9955
6.25	8.5742	7.3093	6.1572
6.50	8.7111	7.4557	6.3207
6.75	8.8491	7.6036	6.4860
7.00	8.9883	7.7530	6.6530
7.25	9.1286	7.9038	6.8218
7.50	9.2701	8.0559	6.9921
7.75	9.4128	8.2095	7.1641

To use the table to find the monthly payment for a 7% loan for 30 years, we see the **payment factor** per $1000 is 6.6530. Then, since the numbers in the table are given as per $1000, to determine the monthly payment for a $120,000 loan, we multiply the 6.6530 by $120,000 and divide by $1000. This tells us the monthly payment is $798.36.

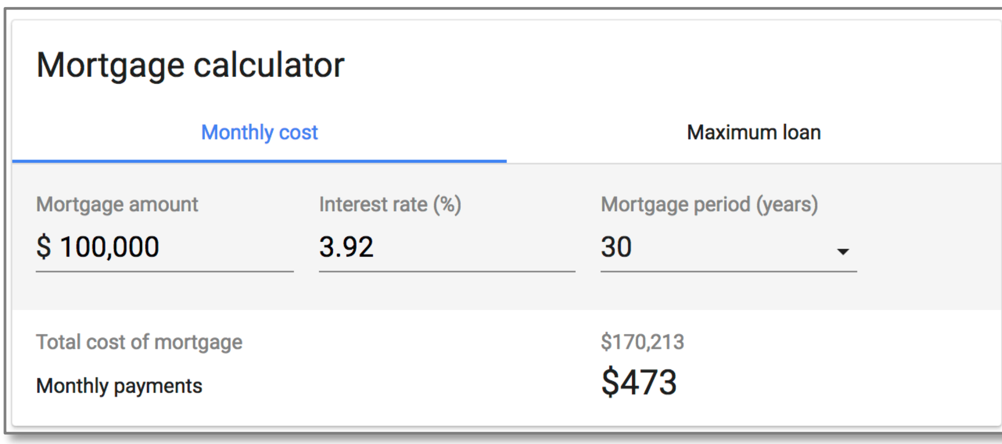
Example 1: Use the amortization table to find the monthly mortgage payment for a $135,000 loan at 5.5% for 30 years.

Using the table, we find the payment factor of 5.5% for 30 years to be 5.6779.
5.6779 × $135,000/$1000 = $766.5165, which has to be rounded to $766.52.

Unfortunately, amortization tables are often incomplete and even hard to locate. Alternatively, we could use a formula and a spreadsheet to determine and compare the monthly payments on a loan with different terms. But, to be honest, the formula is not very simple. Besides, there is a MUCH better option.

Using Online Mortgage Calculators

This leads us to what can easily be considered one of the most useful tools we will ever encounter: Online Mortgage Calculators. Much like the measure converters, a quick **Google** search for "Mortgage Calculator" will yield millions of results in a fraction of a second and, conveniently enough, Google actually embeds a simple mortgage calculator in the results.



Using the embedded Google calculator is extremely simple. The one thing worth noting is the interest rate is entered as a percent, not a decimal. With that in mind, we enter the loan amount and interest rate, and then choose the term from the period drop-down options. At the bottom we will automatically see two numbers. The monthly payment amount and the total cost of the mortgage.

The total cost number is only accurate if our monthly payments are exactly as listed. If we make a payment larger than the indicated amount, the loan will be paid off faster and we will actually save a bit of money in the end. Also, the calculator rounds those amounts to the nearest whole dollar. For a $100,000 loan at 3.92% for 30 years, the payment is $472.81, with the total cost of the mortgage being $170,213.30. Since our goal is general approximations, we will use the embedded calculator and round our amounts to the nearest whole dollar. It is also worth noting that, because the calculator rounded $472.81 to $473, the total cost doesn't match 360 payments of $473 ($473×360 = $170,280.) In other words, if we are working with the total cost of the mortgage, use the rounded number from the calculator.

Example 2: Use the embedded Google mortgage calculator to find the monthly mortgage payment for a $135,000 loan at 5.5% for 30 years.

Pull out your phone or jump online and do a Google search for "mortgage calculator." In the calculator embedded in the results, enter 135,000 for the amount, 5.5 for the rate and set the period to 30 years. The payment appears at the bottom of the calculator: $767. Really, that's it!

With the simple google calculator, we can quickly see the difference in monthly payments when we consider different loan options.

Example 3: Mason wishes to borrow $180,000, and he qualifies for a rate of 5.00%. Find the monthly payment for this loan if the term is 30 years. By how much will the payment go up if he reduces the loan to 20 years? Round your answers to the nearest whole dollar.

Using the Google calculator,
$180,000 at 5% for 30 years gives a monthly payment of $966, and
$180,000 at 5% for 20 years gives a monthly payment if $1188.
The increase is $222.

Why would Mason want a higher monthly payment? Paying the loan off quicker means paying less interest over the life of the loan. For the 30-year loan, the total cost of the mortgage (from the calculator) is $347,860. For the 20-year loan, the total cost is $285,101. So, for paying a couple hundred extra each month, he would save $62,759 AND pay off the loan 10 years earlier.

The Complete Monthly Payment

The monthly payment involved with a loan amount will constitute the majority of the cost of home ownership, but there are other costs that must be considered as well. Probably the most important of all topics in this section is the calculation of the COMPLETE monthly payment.

Real estate taxes are computed as a percent of the assessed value of the home, not the loan amount. Even though they are usually due in quarterly installments, the lending institution is willing to collect 1/12 of the annual taxes each month, put the designated amount into an escrow account (which allows them to collect interest!), and then pay the bills when they come due. Once the loan is paid off, the homeowner becomes responsible for paying those taxes directly to the appropriate governmental body, such as the County Assessor's Office.

Even though we can legally own a home without **homeowner's insurance**, the importance of insurance cannot be underestimated. A home is much too important and much too large of an investment to go without insurance. No one plans on having water leaks, trees being toppled by windstorms, or dealing with fire damage. Rates and quality of coverage can vary greatly from insurance company to insurance company, but the cost of repairs can quickly eclipse the amount paid for insurance. If we borrow money from a lending institution to purchase a home, that institution may require us to obtain homeowner's insurance until we pay off the loan. And, like taxes, the lending institution is willing to collect 1/12 of your annual insurance premiums and pay the bills on your behalf. Acting as the intermediary between the homeowner and the insurance company allows the lending institution to ensure their investment is protected.

Home Owner's Association (HOA) dues are paid for the convenience of living in some neighborhoods. These payments cover things like the costs associated with neighborhood security patrols, maintaining parks, landscaping, and street cleaning. They can be higher for gated communities and condominiums, and like taxes and insurance, the lending institution may serve as an intermediary in making these payments by collecting 1/12 of the annual dues and paying the bills when they come due. And, also like taxes and insurance, the homeowner becomes solely responsible for these payments once the loan is paid off. Neglecting or refusing to pay these dues can result in substantial fines and even having a lien placed on the home.

Next, if the outstanding principal on a loan is more than 80% of the value of the home, **private mortgage insurance** (PMI) may also be required. PMI allows the lending institution to collect insurance money if the borrower stops paying on the loan. In such a case, the institution would also foreclose on the home and sell it to someone else, as well. If the buyer can pay 20% of the home's value at the time of purchase, there is no need to get PMI. If the buyer is forced to pay PMI on a loan, once the loan amount is less than 80% of the value of the home (by payments and home value increases), the loan would need to be refinanced to stop the collection of PMI. PMI fees are usually 1-2% of the loan amount and can vary depending on the borrower's credit score. In our case, to simplify our calculations a little bit (and since it is not required of every homeowner), we will not deal with PMI.

Example 4: Assuming a loan requires the inclusion of real estate taxes and home owner's insurance, find the complete monthly mortgage payment for the following. Round your answer to the nearest whole dollar.

Loan: $105,000 at 5.9% for 30 years
Assessed Value of the House: $140,000
Annual Real Estate Taxes: 2.5% of Assessed Value
Homeowner's Insurance: $480 per year

Using the Google calculator, the loan payment is found to be $623.
Taxes: (0.025)($140,000) = $3500 per year, which adds $292 per month to the payment.
Insurance: $480 per year, will add $40 per month to the payment.

So, our total monthly payment will be $623 + $292 + $40 = $955.

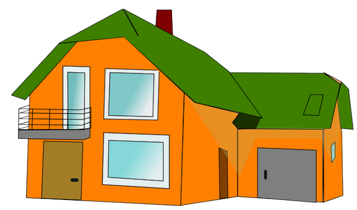
Example 5: Find the complete monthly payment for a $175,000 mortgage loan at 4.5% for 20 years. The assessed value of the home is $240,000. The annual taxes on the home are 1.5% of the assessed value and the insurance on the home costs $600 per year. Round your answer to the nearest dollar.

Using the amortization table, the payment factor is 6.3265, which makes the mortgage payment $1107 per month. The taxes are (0.015)($240,000) = $3600 per year, which adds to $300 per month. The monthly insurance payment is $50.

Thus, the complete monthly payment is $1457.

Affordability Guidelines

When we go to buy a home, the very first question should be "What can we afford?" There are a few traditional guidelines that help consumers determine how much they should spend on housing costs. These guidelines can vary from institution to institution, but the general framework remains the same.



The Affordability Guidelines presented in this book are examples of guidelines used by lending institutions. They are intended to give the prospective home buyer a practical idea of what can be afforded. Different institutions may have similar guidelines which may use slightly different factors. It's their money and they get to decide to whom they will lend it.

Guideline #1: The amount of the mortgage loan should not exceed three times the borrower's annual gross income. Remember, this is just a consideration of the cost of the home and is not a Guideline on the actual monthly payment.

Example 6: If Rusty makes $75,000 per year, how much of a loan will the bank give him to buy a home?

$3 \times \$75{,}000 = \$225{,}000$

The next two guidelines deal with the amount of a monthly budget that can be devoted to housing expenses. Recalling our discussion on monthly budgets, if we have extra significant debt obligations (car payments, student loans, unpaid credit card balances, etc.), housing expenses should be limited to 25-30% of our monthly income. If those extra debt obligations do not exist, our housing expenses can go as high as 40%.

Guideline #2: If a family has other significant monthly debt obligations, such as car payments, credit cards, or student loans, the family's monthly housing expenses, including mortgage payment, property taxes, and private mortgage insurance, should be limited to no more than 25% of their monthly gross income (income prior to deductions).

Taking the first two guidelines into account, a family with an annual gross income of $60,000, should not be given a home loan of more than $180,000, while the monthly expenses for housing should not exceed ($60,000)×(1/12)×(0.25) = $1250.

Realistically, however, the factor that determines whether or not a buyer can afford a home is whether or not they can obtain a loan. Most of us don't have $180,000 lying around, so we have to borrow money. If a prospective buyer has good credit scores and no other significant monthly debt obligations, many banks will allow for 35-40% of the borrower's monthly income to be devoted to housing expenses. For our purposes, we will use 38%.

Guideline #3: If a family has no other significant monthly debt obligations, the monthly housing expenses can be as high as 38% of their gross monthly income.

Returning to our family that makes $60,000 per year, if they have no other significant debt obligations, Guideline #3 says they can afford monthly housing expenses of ($60,000)×(1/12)×(0.38) = $1900.

When following the listed Affordability Guidelines to determine the maximum monthly payment for housing expenses, be sure to note if extra monthly debt obligations exist.

If the extra debt exists, use Guideline #2 (25%). If not, use Guideline #3 (38%).

Example 7: John Doe has gross monthly income of $5125. He does have a car payment and unpaid credit card debt. According to the Affordability Guidelines in this text, how much of his monthly income can he devote to housing expenses?

We don't need to dollar amounts for those extra payments; we just need to know they exist to steer us to Guideline #2.

Monthly Housing Expenses: ($5125)(0.25) = $1281.25.

Once again, the factors of 25% (in Guideline #2) and 38% (in Guideline #3) are just for illustrative purposes in this text and may vary from institution to institution. They are, however, representative of the practices used in determining whether or not a bank will lend money to a prospective home buyer.

Section 3.5 Exercises

For Exercises #1-4, use the amortization table presented earlier in this section to find the monthly mortgage payment for loans with the following conditions. Round your answers to the nearest whole dollar and verify them using an online mortgage calculator.

1. \$150,000 for 20 years at 5.25%

2. \$172,000 for 30 years at 7.00%

3. \$210,000 for 30 years at 5.75%

4. \$275,000 for 15 years at 5.50%

5. Dave wishes to borrow \$210,000, and he qualifies for a rate of 7.25%. Find the monthly mortgage payment for this loan if the term is 30 years. By how much will the payment change if he reduces the loan to 20 years? Round your answers to the nearest whole dollar.

6. Trevor wishes to borrow \$280,000, and he qualifies for a rate of 5.50%. Find the monthly mortgage payment for this loan if the term is 30 years. By how much will the payment change if he reduces the loan to 15 years? Round your answers to the nearest whole dollar.

7. Laura wishes to borrow \$175,000, and she qualifies for a rate of 6.25%. Find the monthly mortgage payment for this loan if the term is 20 years. By how much will the payment change if she takes out the loan to 30 years? Round your answers to the nearest whole dollar.

8. Find the complete monthly payment for a \$275,000 mortgage loan at 5.5% for 30 years. The assessed value of the home is \$300,000. The annual taxes on the home are 1.2% of the assessed value and the insurance on the home costs \$756 per year. Round your answers to the nearest whole dollar.

9. Find the complete monthly payment for a \$205,000 mortgage loan at 5.25% for 20 years. The assessed value of the home is \$240,000. The annual taxes on the home are 1.25% of the assessed value and the insurance on the home costs \$456 per year. Round your answers to the nearest whole dollar.

10. Find the complete monthly payment for a \$185,000 mortgage loan at 4.5% for 30 years. The assessed value of the home is \$200,000. The annual taxes on the home are 0.9% of the assessed value and the insurance on the home costs \$372 per year. Round your answers to the nearest whole dollar.

11. According to the Affordability Guidelines, if a family has an annual income of $62,000, what should be the maximum amount of a mortgage loan the family could afford?

12. According to the Affordability Guidelines, if a family has a gross annual income of $62,000, and other significant debt obligations, how much of their monthly income can be devoted to housing expenses? Round your answers to the nearest whole dollar.

13. According to the Affordability Guidelines, if a family has a gross annual income of $122,000, what should be the maximum amount of a mortgage loan the family could afford?

14. According to the Affordability Guidelines, if a family has a gross annual income of $33,000, and no other significant debt obligations, how much of their monthly income can be devoted to housing expenses? Round your answers to the nearest whole dollar.

15. According to the Affordability Guidelines, if a family has a gross annual income of $174,000, and other significant debt obligations, how much of their monthly income can be devoted to housing expenses? Round your answers to the nearest whole dollar.

16. According to the Affordability Guidelines, if a family has a gross annual income of $48,000, and other significant debt obligations, how much of their monthly income can be devoted to housing expenses? Round your answers to the nearest whole dollar.

17. According to the Affordability Guidelines, if a family has a gross annual income of $120,000, and no other significant debt obligations, how much of their monthly income can be devoted to housing expenses? Round your answers to the nearest whole dollar.

18. According to the Affordability Guidelines, if a family has a gross annual income of \$312,000, what should be the maximum amount of a mortgage loan the family could afford?

19. According to the Affordability Guidelines, if a family has a gross annual income of \$57,000, and no other significant debt obligations, how much of their monthly income can be devoted to housing expenses? Round your answers to the nearest whole dollar.

Section 3.5 Exercise Solutions

1. Use 6.7384 as the payment factor. (6.7384)($150,000)/1000 = $1011

2. Use 6.6530 as the payment factor. (6.6530)($172,000)/1000 = $1144

3. Use 5.8358 as the payment factor. (5.8358)($210,000)/1000 = $1226

4. Use 8.1708 as the payment factor. (8.1708)($275,000)/1000 = $2247

5. $210,000 at 7.25% for 30 years is $1433. $210,000 at 7.25% for 20 years is $1660. The payment will go up $227.

6. $280,000 at 5.5% for 30 years is $1590. $280,000 at 5.5% for 15 years is $2288. The payment will go up $698.

7. $175,000 at 6.25% for 20 years is $1279. $175,000 at 6.25% for 30 years is $1078. The payment will go down $201.

8. The monthly loan payment for $275,000 at 5.5% for 30 years is $1561. The monthly taxes are (0.012)($300,000)/12 = $300. The monthly insurance premium is $756/12 = $63. That makes the complete monthly payment $1561 + $300 + $63 = $1924.

9. The monthly loan payment for $205,000 at 5.25% for 20 years is $1381. The monthly taxes are (0.0125)($240,000)/12 = $250. The monthly insurance premium is $456/12 = $38. That makes the complete monthly payment $1381 + $250 + $38 = $1669.

10. The monthly loan payment on $185,000 at 4.5% for 30 years is $937. The monthly taxes are (0.009)($200,000)/12 = $150. The monthly insurance premium is $372/12 = $31. That makes the complete monthly payment $937 + $150 + $31 = $1118.

11. We are looking for the maximum loan amount, so use Guideline #1. 3($62,000) = $186,000

12. We are looking for monthly housing expenses and other debt exists. Use Guideline #2. Be sure to divide the annual income by 12. (0.25)($62,000)/12 = $1291.666… → $1292

13. We are looking for the maximum loan amount, so use Guideline #1. 3($122,000) = $366,000

14. We are looking for monthly housing expenses and other debt does not exist. Use Guideline #3. Be sure to divide the annual income by 12. (0.38)($33,000)/12 = $1045

15. We are looking for monthly housing expenses and other debt exists. Use Guideline #2. Be sure to divide the annual income by 12. (0.25)($174,000)/12 = $3625

16. We are looking for monthly housing expenses and other debt exists. Use Guideline #2. Be sure to divide the annual income by 12. (0.25)($48,000)/12 = $1000

17. We are looking for monthly housing expenses and other debt does not exist. Use Guideline #3. Be sure to divide the annual income by 12. (0.38)($120,000)/12 = $3800

18. We are looking for the maximum loan amount, so use Guideline #1. 3($312,000) = $936,000

19. We are looking for monthly housing expenses and other debt does not exist. Use Guideline #3. Be sure to divide the annual income by 12. (0.38)($57,000)/12 = $1805

CHAPTER 3 CREDITS

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CHAPTER 4: PROBABILITY & STATISTICS

What's the difference between **probability** and **statistics**? In simplest terms, to describe a whole set of objects by looking at a few specific items randomly selected from the entire group, we use statistics. To attempt to predict the results of one particular event based on information that describes an entire set, we use probability.

Randomness is extremely important to the worlds of probability and statistics. If the systems under consideration were not random, they would be predictable. If they were predictable, then there would not be a need to study them like we do. Thus, to keep things truly random, unless otherwise stated, we will assume all events we study are random in nature and all outcomes are equally likely. We further assume, in this book, all dice and coins are fair and balanced, all decks of cards are well shuffled, and all balls in urns have been thoroughly mixed… unless otherwise stated.

CHAPTER OUTLINE AND OBJECTIVES

Section 4.1: Counting, Factorials, Permutations & Combinations

A. Understand different counting techniques.
B. Be able to compute factorials, permutations and combinations.
C. Understand counting features on your calculator.

Section 4.2: Simple Probability & Odds

A. Understand basic probability.
B. Be able to compute probabilities using dice and cards.
C. Be able to compute the odds of a specific event.

Section 4.3: Fair Price & Expected Value

A. Be able to determine the fair price of a game.
B. Be able to compute the expected value of an event.

Section 4.4: Averages

A. Be able to compute the mean, median and mode for a set of data.
B. Be able to compute a grade point average (GPA).

Section 4.5: Charts & Graphs

A. Be able to organize data in a meaningful way.
B. Understand the appropriate uses of bar graphs, line graphs and pie charts.
C. Be able to create bar graphs, line graphs and pie charts.

4.1: COUNTING, FACTORIALS, PERMUTATIONS & COMBINATIONS

Counting

Counting. Seems simple, right? We know how to count, or we wouldn't have made it into this math class. Well, yes and no. What we're going to examine are some different situations that require different types of counting.

Let's imagine we've gone into a restaurant, and the menu looks like this:

- Appetizers: Breadsticks, Salad
- Main Dishes: Lasagna, Chicken, Meatloaf
- Desserts: Pie, Ice Cream

You are instructed by the waiter to choose exactly one item from each column to create your meal. How many different meals are possible?

One approach we can take in trying to answer this question is to make a list of all the possible meals. If we're going to do this, we should be as systematic as possible, so that we don't leave out any of the meals.

Going across the top row, the first meal that we could make would be: Breadsticks-Lasagna-Pie (we will abbreviate this as B-L-P). We could also have Breadsticks-Lasagna-Ice Cream (B-L-I). This takes care of every possibility that includes Breadsticks and Lasagna.

So far, we have:

B-L-P and B-L-I

Continuing, we will create every meal that has breadsticks as the appetizer:

B-C-P, B-C-I, B-M-P, B-M-I

That's it. That's every meal that has breadsticks as the appetizer. Now we must consider salad as the appetizer:

S-L-P, S-L-I, S-C-P, S-C-I, S-M-P, S-M-I

Whew. Well, there they are; all twelve possible meals. Remember, though, we were not asked to find the actual meals; we were just asked to find the *number* of meals. If you're thinking, "there MUST be a better way to count them," you're right. It is known as the **Fundamental Counting Principle**.

Fundamental Counting Principle

Find the number of choices for each option and multiply those numbers together.

When we look at our menu from above, we had two choices of appetizer, three choices of main dish, and two choices of dessert. Multiplying these together gives us $2 \times 3 \times 2 = 12$ possible meals. Yep, it really is that easy.

Factorials

Imagine a class with six students in it, and their names are Red, Yellow, Green, Blue, Indigo, and Violet. We're going to line the students up in a single row for a class photograph. How many different photographs are possible? We will apply the Fundamental Counting Principle to determine how many different lineups are possible.

How many choices do we have for the first spot? Well, no students have been used yet, so we have all six to pick from. We have six choices for the first spot.

Now that we have placed a student (any one of them, it really doesn't matter which one) in the first spot, there would be five left to pick from for the second spot. Do you see the Fundamental Counting Principle at work here?

Continuing with the same logic, we can see there are four students available for the third place, 3 for the next one, then two students and, finally, one student will be left for the last place.

We multiply these all together and wind up with $6\times5\times4\times3\times2\times1 = 720$ different ways for the students to line up. If this number seems too large, start listing them out, starting with RYGBIV, then RYGBVI, and so on. You'll probably get tired and accept the 720 as correct, but you just might be able to convince yourself.

What if we were asked the same question about a deck of cards? How many different ways can we place the cards down in a straight line? Well, there are 52 cards, so that will be: $52\times51\times50\times$... WAIT A MINUTE! We don't want to write that all the way down to 1, and doing all that multiplying on a calculator would be hard to do without making a mistake. "There's got to be a better way to write this," you say, and you would be correct. It is known as a **factorial**.

The factorial symbol (commonly known as the exclamation point) gives us a shorthand way to write such problems down, and if we have this symbol on our calculator, it can save us a ton of work. By definition, n! is the product of all the numbers from n down to 1. 4! (pronounced "four factorial") = $4\times3\times2\times1 = 24$.

Factorials get really large really fast.

$1! = 1$
$2! = 2\times1 = 2$
$3! = 3\times2\times1 = 6$
$4! = 4\times3\times2\times1 = 24$
$5! = 5\times4\times3\times2\times1 = 120$
$6! = 6\times5\times4\times3\times2\times1 = 720$
$7! = 7\times6\times5\times4\times3\times2\times1 = 5040$
$8! = 8\times7\times6\times5\times4\times3\times2\times1 = 40{,}320$
$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1 = 362{,}880$
$10! = 10\times9\times8\times7\times6\times5\times4\times3\times2\times1 = 3{,}628{,}800$
$11! = 11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1 = 39{,}916{,}800$
$25! = 15{,}511{,}210{,}043{,}330{,}985{,}984{,}000{,}000$

Remember the question about the different arrangements for a standard deck of cards? Well, 52! is greater than 8.0658×10^{67}, which is a 68-digit number. For comparison, the number one hundred trillion – 100,000,000,000,000 - only has 15 digits!

Essentially, applying a factorial is just another operation, and, in terms of the order of operations factorials are groups of multiplication, so they should be done on the same level as grouped operations that normally appear in parentheses. Alternatively, if we expand the factorials into the corresponding multiplication, we can perform all the multiplications and divisions on the same level and cancel common factors. Do not, however, attempt to cancel numbers while they are still in the factorial notation, unless, of course, they are identical.

Example 1: Compute 9!/3!

$$\frac{9!}{3!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1} = \frac{9\times8\times7\times6\times5\times4}{1} = 60{,}480$$

In the previous example, instead of expanding it to 3×2×1, we could have left 3! in both the numerator and denominator and cancelled the identical factors. We could not, however, have started the problem by cancelling before we expanded the factorials.

COMMON MISTAKE

Do not cancel numbers within factorial notations; expand the factorials first.
Identical factorials appearing as common factors can be cancelled without expanding them.

Our discussion of factorials would not be complete without a couple more pieces of the puzzle.

First, let's talk about 0!. Although it is very tempting to think that 0! = 0, that is incorrect. 0! is actually equal to 1. Some authors will make this claim "out of convenience" and others will claim "as a definition." If you wish to accept 0! = 1 without condition, so be it. If you would like a little proof, think of the following.

> $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$ *and* $(n-1)! = (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$
>
> Then, substituting (n-1)! into the equivalent part of the first equation, we have: $n! = n \times (n-1)!$
>
> Dividing both sides of the equation by n gives us: $n!/n = (n-1)!$
>
> Now, if we let n = 1, the equation becomes: $1!/1 = (1-1)!$
>
> Finally, some minor simplification on both sides gives us: 1 = 0!, which is the same as 0! = 1.

Next, since we have covered factorials for 0 and the positive integers (aka the Whole Numbers), one might be tempted to ask about factorials for fractions, decimals or even negative numbers. Well, the good news there is factorials are only defined for Whole Numbers. ☺

Permutations

A **permutation** is an arrangement of distinct objects that are placed in a specific order. Just like in our class photo example from earlier in the section, if the order changes, then we have a new permutation. We worked with this idea in our photograph example and that helped explain factorials. The photo RGYBIV is different from the photo BIVGYR. The letters are the same, but the order has changed, creating a different photograph. Since changing the order makes the photo different, we are dealing with a permutation. For a permutation, the order matters.

As long as individual objects are not being duplicated, the formula for finding the number of permutations is as follows, where n is the total number of objects, and r is the number of objects being used. The phrase, "being used" may not make sense yet, but it will as we continue to practice.

$$nPr = \frac{n!}{(n-r)!}$$

Although many people resort to memorizing that formula, it makes a bit more sense if we take a few moments to understand it.

Permutations are all the possible ways of doing something. Take another look at our photo example. What if we wanted only four of the six students in the picture? How many ways could it be done? Formally, this is called a "Permutation of 6 things, taken 4 at a time," or 6P4.

From the Fundamental Counting Principle, we have 6 choices for the first person, 5 for the second, 4 for the third, and 3 choices for the last student. That makes 6×5×4×3 = 360 different photos.

Alternatively, if we begin by considering all the possible arrangements with six students, we have 6!. Then, because we have two students left out of the photo, we would need to cancel the possible arrangements of those students. With numbers, this looks like 6!/2!. Thus, the 2×1 would cancel, leaving 6×5×4×3 in the numerator.

Relating that to the formula, the n! in the numerator refers to all possible arrangements. For the denominator, if we have n items and only want to count r of them, then there are (n – r) of them we don't want to count. The (n – r)! in the denominator refers to the arrangements that are not counted.

Now, going back to the class picture example, 6P4 = 6!/(6-4)! = 6!/2! = (6×5×4×3×2×1)/(2×1) = 6×5×4×3 = 360. There are 360 different ways to take a photo using four of the six different students.

Example 2: Compute 7P3.

7P3 = 7!/(7-3)! = 7!/4!

Since we can foresee a 4! cancelling, we won't bother expanding it.
7!/4! = 7×6×5×4!/4! = 7×6×5 = 210

Combinations

A **combination** is a collection of distinct objects in which the order makes no difference. Distinguishing between combinations and permutations can be a little tricky, but remember, for combinations, order does not matter.

Let's suppose, for example, of six students in a class, four of them will be chosen to go on a field trip. Here, we are making a collection of four students, and the order in which they are chosen makes no difference. Each student is either chosen, or not, but it doesn't matter which one is chosen first. While RGBY would make a different photo than GBRY (which indicates a permutation), having RGBY go on the field trip is exactly the same as having GBRY go on the field trip (which indicates a combination). The distinction between the two can be difficult. Try to decide if the order would make a difference. If not, then we have a combination.

In terms of combinations, RGBY is the same as GBRY. In fact, any group of those four students is the same. They are all redundant because order does not matter. Furthermore, in this situation, there are 4! = 24 redundancies.

Finding the number of combinations starts with finding the number of permutations, and then dividing by the number of redundancies.

Just like in the formula for permutations, as long as individual items are not reused, the formula for determining the number of combinations is as follows, where n is the total number of objects, and r is the number of objects being used.

$$nCr = \frac{n!}{(n-r)!r!}$$

Think about that formula for a minute. Just like with the formula for permutations, n! represents the total number of all possible arrangements and (n – r)! has us cancelling out the unused arrangements. Then, the extra r! in the denominator has us cancelling out the redundancies. It may also be helpful to remember, since we cancel out the redundancies, the number of combinations for a situation can never be more than the number of permutations for the same number of items.

Revisiting the students being chosen for our field trip, let's say we want to pick four out of six students. Here, since we have six students, but only four of them will be chosen, we have a "combination of 4 things chosen from a group of 6," which is 6C4.

Using the formula, we get:

$$6C4 = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!} = \frac{6\times5\times4!}{2\times1\times4!} = \frac{6\times5}{2\times1} = 15$$

There are 15 different ways to pick four of the six students to take on the field trip.

A key thing to notice in working with the formula for combinations is the factorials in the denominator are not multiplied together to form a single factorial. Combining the 4! and the 2! to make 8! would be incorrect. Don't do that.

Notice that the numbers used in the combination example are the same as those used in the example for the permutation, but the answers are VERY different.

Finally, in regard to the notation, sometimes we indicate permutations and combinations by subscripting the n and the r, as in $_nP_r$ and $_nC_r$. And, in some texts, you may see these written as P(n, r) and C(n, r), respectively. For our purposes, we will just write nPr and nCr without subscripting the n and r. In all cases, though, the P and C are capital letters, and the n and r are lower case letters that are usually replaced with numbers.

Example 3: A job has 10 applicants and 3 of them will be picked for an interview. In how many ways can this be done?

Notice that there is no mention of order here. Picking applicants #3, #5 and #9 is the same as picking #9, #3 and then #5. Either way, the same three applicants will get interviewed. Since order does not matter, we have a combination.

10C3 = $\frac{10!}{(10-3)!3!} = \frac{10\times9\times8\times7!}{7!3!} = \frac{10\times9\times8}{3\times2\times1}$ = 120

There are 120 different ways to pick three applicants out of a pool of ten.

The Difference

Often, the hardest part about working with permutations and combinations is determining which one to use - based on the description of the situation. Sometimes, certain key words can help us decide if we have a permutation or a combination.

Key Words:

- Permutations: Arrangement, Order
- Combinations: Choose, Pick, Select

In general, ALWAYS ask yourself, "Does order matter." If so, we have a permutation. If not, it is a combination.

When trying to decide if a situation describes a permutation or a combination, ask yourself if order matters. If so, it is a permutation. If not, it is a combination.

Also think about the logistics of the situation. If we have a box of 50 photographs and we want to pick 15 of them to put into an album, we are working with a combination because it does not matter in what order we choose them – we just have to pick 15 of them. If, however, we are actually putting the photos in the photo album, the order makes a difference. Putting a picture of your mother on the first page as opposed to the tenth page makes a difference. If the order matters, we have a permutation.

Using Your Calculator

If you have a scientific calculator, you may have a factorial key on it. On many calculators, we type in the number, and then look for a button labeled “x!” or “n!”. On some calculators, once you hit the factorial button, the computation will happen right away – without even having to hit the = key. On other calculators, you will have to hit =.

If your calculator can perform factorials, then it is likely it can also perform permutations and combinations, the keys will be labeled “nPr” and “nCr,” respectively. Usage of these functions varies from calculator to calculator. In most cases, you type in the value for n, hit the nPr (or nCr) key, then type in the value for r, and hit the = key. If this sequence of entries does not work on your calculator, consult your calculator’s owner’s manual.

Do keep in mind; even if your calculator performs factorials, permutations and combinations, you still need to be able to recognize which of those functions applies to the given situation. Remember, the calculator is only as accurate as the person using it.

Section 4.1 Exercises

1. A fraternity is to elect a president and a treasurer from the group of 40 members. How many ways can those two officers be elected?

2. Sandra has 9 shirts and 5 pairs of pants. Assuming that everything matches, how many different outfits can she make?

3. If a special, three-character code consists of a capital letter followed by two numbers, how many different codes are possible?

4. How many three-digit numbers are there?

5. Prior to 1995, three-digit telephone area codes were not allowed to use a 0 or a 1 for the first digit, the middle digit had to be a 0 or a 1, and the third digit could not be the same as the middle digit. With those rules, how many area codes were possible?

6. A school gymnasium has 6 different doors. How many ways can a person enter the gymnasium and then leave through a different door?

7. If a license plate consists of 3 letters followed by 3 numbers, how many of these license plates are possible?

8. If a license plate consists of 3 letters followed by 3 numbers AND repetition is not allowed, how many license plates are possible?

For Exercises #9-16, evaluate.

9. 4!

10. 7!

11. 11!

12. 9!/5!

13. 7P3

14. 5P5

15. 5P1

16. 9P5

17. You have 8 books to place on a shelf, but you only have room for 5 of them. How many different ways can you arrange books on this shelf?

18. Sam has 5 favorite football teams, and every week he puts their flags on his flagpole in random order. How many different ways can he arrange the flags?

19. Sam has 5 favorite football teams, and every week he puts their flags on his flagpole in random order. If he only has room for 3 of the flags on his flagpole, how many different ways can they be arranged?

20. A baseball batting order is made up of 9 players. How many different batting orders are possible?

21. Ten runners are in a race. How many different ways can 1st, 2nd, and 3rd place be determined?

For Exercises #22-25, evaluate.

22. 7C4

23. 8C3

24. 5C5

25. 6C0

26. Of seven students in a class, two will be chosen to go on a field trip. How many different ways can they be selected?

27. Of the 15 players at an awards dinner, 3 of them will be given identical trophies. How many different ways can the trophies be given out?

28. A teacher chooses 5 of her 12 students to help clean the room after school. In how many ways can the students be chosen?

29. A student must answer 5 of the 9 essay questions that are on an exam. How many ways can the student select 5 questions?

Section 4.1 Exercise Solutions

1. $(40)(39) = 1560$

2. $(9)(5) = 45$

3. $(26)(10)(10) = 2600$

4. It's tempting to say $(10)(10)(10) = 1000$. But remember, we do not start numbers with a 0. That means there are 9 choices for the first digit and 10 for each of the other two. There are $(9)(10)(10) = 900$ three-digit numbers.

5. $(8)(2)(9) = 144$

6. $(6)(5) = 30$

7. $(26)(26)(26)(10)(10)(10) = 17{,}576{,}000$

8. $(26)(25)(24)(10)(9)(8) = 11{,}232{,}000$

9. $4! = 4\times3\times2\times1 = 24$

10. $7! = 7\times6\times5\times4\times3\times2\times1 = 5040$

11. $11! = 11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1 = 39{,}916{,}800$

12. $9!/5! = 9\times8\times7\times6\times5!/5! = 9\times8\times7\times6 = 3024$

13. $7P3 = 7!/(7-3)! = 7!/4! = 7\times6\times5 = 210$

14. $5P5 = 5!/(5-5)! = 5!/0! = 5!/1 = 120$

15. $5P1 = 5!/(5-1)! = 5!/4! = 5$

16. $9P5 = 9!/(9-5)! = 9!/4! = 9\times8\times7\times6\times5 = 15{,}120$

17. The use of the word "arrange" indicates it is a permutation. $8P5 = 6720$

18. The use of the word "arrange" indicates it is a permutation. $5P5 = 120$

19. The use of the word "arrange" indicates it is a permutation. $5P3 = 60$

20. Switching just two batters makes a different lineup. This is a permutation. $9! = 362{,}880$

21. $(10)(9)(8) = 720$

22. $7C4 = 7!/[(7-4)!4!] = 7!/[3!4!] = [7\times6\times5\times4!]/[3\times2\times1\times4!] = 210/6 = 35$

23. $8C3 = 8!/[(8-3)!3!] = 8!/[5!3!] = [8\times7\times6\times5!]/[5!\times3\times2\times1] = 336/6 = 56$

24. $5C5 = 5!/[(5-5)!5!] = 5!/[0!5!] = 5!/5! = 1$

25. $6C0 = 6!/[(6-0)!0!] = 6!/6! = 1$

26. Order does not matter, so it is a combination.
$7C2 = 7!/[5!2!] = [7\times6]/[2\times1] = 21$

27. Order does not matter, so it is a combination.
$15C3 = 15!/[12!3!] = [15\times14\times13]/[3\times2\times1] = 455$

28. Order does not matter, so it is a combination.
$12C5 = 12!/[7!5!] = [12\times11\times10\times9\times8]/[5\times4\times3\times2\times1] = 792$

29. Order does not matter, so it is a combination.
$9C5 = 9!/[4!5!] = [9\times8\times7\times6]/[4\times3\times2\times1] = 126$

4.2: SIMPLE PROBABILITY & ODDS

Terminology

A **sample space** is the list of everything that could possibly happen during an experiment, such as flipping a coin. It is common to list the sample space as a set, and the sample space for flipping a coin would be relatively short: {heads, tails}, or to abbreviate: {H, T}. Each result of an experiment is called an **outcome**, and outcomes can be listed individually or collectively as **events**.

Thus, if we are rolling a single six-sided die and are interested in the number of dots on the top side, our sample space, which contains six different outcomes, would be: {1, 2, 3, 4, 5, 6} then an example of an event would be, "rolling a three" which would correspond to the set: {3}. Another example of an event would be, "rolling an even number," which would correspond to the set: {2, 4, 6}.

There are two different kinds of probability. More formal definitions are out there, but what it boils down to is two different situations: Making predictions on what should happen, and making predictions based upon what has already happened.

Theoretical Probability

This is what should happen. Flipping a coin, we would expect heads to occur half of the time. If we flipped a coin 100 times, we would expect to get about 50 heads. **Theoretical probability** (or just **probability**) is calculated as the likelihood of obtaining a specific event.

Relative Frequency

This is what has already happened. In different publications, we often see this referred to as “empirical probability,” “estimated probability,” or even “experimental probability.” To avoid confusing it with theoretical probability, we will avoid the word “probability” and refer to it as **relative frequency** (or just **frequency**). To discover the relative frequency of an event, we must conduct an experiment or a study, and then use those results to predict future results. If we flipped a coin 100 times and got 97 heads, we might suspect that this coin wasn't fair, and we could use those results to say we are much more likely to get a head on the next flip.

Disclaimers

If it is impossible to predict all possible outcomes, or the outcomes are not equally likely, then probabilities can only be determined from the result of an experiment, and, hence, are actually frequencies. For example: In baseball, it is impossible to determine whether or not a player will get a hit before he goes to bat (the outcomes are not equally likely), but, based on all of his previous at bats, a fair prediction can be made. The player's batting average is, essentially, a relative frequency being used to make a prediction. Thus, based on previous at bats, a player with a batting average of .315 has a 31.5% chance of getting a hit in his next at bat.

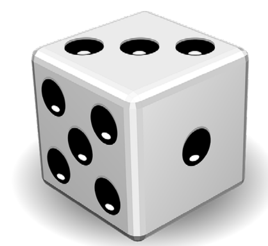
Because some entities use the term “probability” in describing both theoretical probability and relative frequency, it can be difficult to distinguish between the two. In an ideal case, the two will be equal, and it would not matter. After an experiment, if the two are very different, it is usually an indication of an unusual occurrence. In general, the problems that we will be working on will be asking us to find a theoretical probability, and, unless otherwise noted, we also need to assume each outcome has an equal chance of occurring.

Focusing on theoretical probability, if you remember only one thing, let it be that the probability of an even is the ratio of the successes to the total number of outcomes.

Probability = (# of Successes)/(# of Possible Outcomes)

A key thing to note here is that the term "success" is not necessarily something that corresponds to winning. We have a "success" if the event of which we are interested in finding the probability actually occurs. Furthermore, unless directed otherwise, we will write probabilities as fractions.

Example 1: A single six-sided die is to be rolled. What is the probability of rolling a 4? Alternatively, using shorter notation, find P(4).



The number of outcomes when rolling a single die is 6.
That will be the denominator of our answer.

The number of successes (notice, we don't WIN anything), or 4's, on the die is 1.
That will be the numerator of our answer.

So, the probability of rolling a four, or P(4) = 1/6

Example 2: Find the probability of rolling an even number, or P(even).

The number of outcomes when rolling a single die is 6. The number of successes, or even numbers, on the die is 3.

So, the probability of rolling an even number, or P(even) = 3/6.

This answer can be reduced to 1/2, but that is not necessary when working probability problems. The reason for leaving the fraction unreduced is because this answer can provide us information about the raw number of successes and outcomes for the situation we are considering. With P(even) = 3/6, we can tell there were 3 successes out of 6 attempts. If the fraction were reduced to P(even) = 1/2, we would not be able to tell how many successes or outcomes were there originally. We could also represent this answer 0.50 or even 50%. These are not incorrect, but it is more common to write probabilities as fractions, instead of decimals or percents.

Rolling a Pair of Dice

It is very common to examine problems that involve rolling a pair of dice and taking the sum of the faces that are showing upward. There are 6 different outcomes on each die, and we can use the Fundamental Counting Principle to find that there will be 36 (6×6) different outcomes in this situation.

The table below is a way to list all 36 of those outcomes, with the sum of the numbers showing on top of the dice being at the top of each column in the table. Note: In each pair of numbers, the first number represents the result of rolling the first die, and the second number represents the result of rolling the second die.

2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	
		(3,1)	(3,2)	(3,3)	(3,4)	(4,4)	(5,4)	(6,4)		
			(4,1)	(4,2)	(4,3)	(5,3)	(6,3)			
				(5,1)	(5,2)	(6,2)				
					(6,1)					

Example 3: Two dice are rolled, and the sum of the faces is obtained. Find the probability that the sum is 9.

The number of successes, or ways to roll a nine, in our list of outcomes is 4. The number of outcomes when rolling a pair of dice is 36.

So, the probability of getting a sum of 9, or P(9) = 4/36.

Deck of Cards

Problems involving a standard deck of cards are very common. There are 52 cards in a standard deck, and they are divided into four suits, as follows:

Black Cards

♠ - Spades: {Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King} ... 13 cards

♣ - Clubs: {Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King} ... 13 cards

Red Cards

♥ - Hearts: {Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King} ... 13 cards

♦ - Diamonds: {Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King} ... 13 cards

Since the Jack, Queen, and King of each suit have pictures of people on them, they are often referred to as "face cards."

Example 4: In selecting one card from a standard deck, find the probability of selecting a Diamond.

The number of successes, or diamonds, in the deck is 13. The number of outcomes when picking a card is 52.

So, the probability of picking a Diamond, or P(♦) = 13/52.

Or vs. And

Answer "Yes" or "No" to the following question:

Counting yourself, is there one or more people in the room right now?

For an "or" event, at least one of the items must be true. Whether you are alone or not is irrelevant; the answer to the question is "Yes." If you are alone, the condition of one person is satisfied. If you are not alone, the condition of "more" is satisfied. Either way, "Yes" is the answer to the question.

For an "and" event, ALL of the listed outcomes must be successes. Yes or No: Are you standing OR sitting right now? Assuming you are not lying down, YES, you are either standing OR sitting right now. Next, Yes or No: Are you standing AND sitting right now? If you said "Yes," exactly *how* are you BOTH standing AND sitting at the same time?

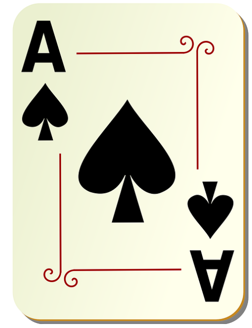
Example 5: In selecting one card from a standard deck, find the probability of selecting a 5 or a Heart.

The new idea here is the presence of the word "or," which tells us that 5s as well as Hearts are now to be considered successes.

The number of outcomes when picking a card is 52. The number of 5s in the deck is 4, but these are not our only successes. When figuring out our numerator, we also must consider the Hearts. There are 13 of them, but we have already counted the 5 of Hearts, so there are 12 Hearts still to be included. Adding all our successes together, we have 16 of them, and that will be the numerator of our answer.

So, the probability of picking a 5 or a Heart, or P(5 or ♥) = 16/52.

Example 6: In selecting one card from a standard deck, find the probability of selecting an Ace and a Spade.



The new idea here is the presence of the word "and," which tells us that a card must be an Ace and a Spade to be considered a success.

The number of outcomes when picking a card is 52. The number of successes, or cards that meet both conditions, is 1. That is, there is only 1 card that is BOTH an Ace AND a Spade.

So, the probability of picking an Ace and a Spade, or P(Ace and ♠) = 1/52.

A Couple of Important Observations

On a roll of a single die, what is P(20)? It can't happen, right? There is no way to roll a 20 on a single die. So, let's see what that would look like as a probability. The number of outcomes is 6, and the number of successes is 0. So, P(20) = 0/6 = 0.

An event that can never happen has a probability of 0.

On a roll of a single die, what is P(number less than 10)? Well, all of the numbers are less than 10, so this will always happen. The number of outcomes is 6, and the number of successes is 6. So, P(number < 10) = 6/6 = 1.

An event that will definitely happen has a probability of 1.

The two previous concepts lead us to a very important conclusion.

All probabilities are between zero and one, INCLUSIVE.

Odds

The **odds in favor** of a particular event are the ratio of the chance of successes to the chance of failures. The **odds against** a particular event are computed as the ratio of chance of failures to chance of successes. Thus, if we know the odds against an event are 10 to 1, in 11 identical attempts, we should expect 10 failures and 1 success of the event in question.

Odds statements can be expressed using the word "to," as a ratio using a colon to separate the two numbers, or even a fraction. To keep things simple, we will state our odds statements as a ratio using a colon to separate the two numbers.

Express odds statements as a ratio using a colon to separate the two numbers.

Example 7: The probability of an event is 21/38. What are the odds against of the event?

Since the p(success) = 21/38, there would 21 successes out of 38 outcomes. This means there would be 38 - 21 = 17 failures.

The odds against this event would be 17:21.

Example 8: If the odds against an event are 3:11, what is the probability of success for the event in question?

From the odds statement, there are 3 failures and 11 successes. This means there must be 14 (3+11) outcomes.

Thus, P(Success) = 11/14.

Example 9: There are 15 red and 25 white gumballs in a machine. If we buy one gumball at random, what are the odds in favor of the gumball being red?

Odds in favor is the ratio of successes to fails. With the “successes” being red gumballs, we see there are 15 successes and 25 fails.

The odds in favor of getting a red gumball are 15:25.

Section 4.2 Exercises

State probability answers as fractions. State odds statements as ratios of two numbers separated by a colon.

For Exercises #1-9, a quarter is flipped, and a single standard die is rolled.

1. Find the probability of getting Heads and a 3.

2. Find the probability of getting Tails and a 7.

3. Find the probability of getting Heads and an odd number.

4. Find the probability of getting Heads and a number greater than 6.

5. Find the probability of getting Tails or a 7.

6. Find the probability of getting Tails or a number less than 9.

7. Find the odds against obtaining a Head.

8. Find the odds against obtaining a 6.

9. Find the odds against obtaining an even number.

For Exercises #10-14, a pair of dice is rolled, and the sum of the faces is obtained.

10. Find the probability that the sum is 6.

11. Find the probability that the sum is odd.

12. Find the probability that the sum is a multiple of 3.

13. Find the odds against the sum being 11.

14. Find the odds against the sum being 5.

For Exercises #15-19, a bag of jellybeans contains 5 red beans, 3 blue beans, 7 orange beans, 4 green beans, 2 yellow beans and 3 purple beans.

15. Find the probability that a jellybean selected at random is orange.

16. Find the probability that a jellybean selected at random is green or yellow.

17. Find the probability that a jellybean selected at random is neither red nor purple.

18. Find the probability that a jellybean selected at random is anything but yellow.

19. Find the probability that a jellybean selected at random is black.

For Exercises #20-28, a single card is drawn from a standard deck of 52 cards.

20. Find the probability that the card is the queen of hearts.

21. Find the probability that the card is a six.

22. Find the probability that the card is a black card.

23. Find the probability that the card is a red nine.

24. Find the probability that the card is an ace or a heart.

25. Find the probability that the card is an ace and a heart.

26. Find the probability that the card is a face card.

27. Find the odds against drawing a king.

28. Find the odds against drawing a queen or a diamond.

29. The odds against an event are 5:2. What is the probability of success for the event?

30. The odds against an event are 3:7. What is the probability of success for the event?

Section 4.2 Exercise Solutions

For Exercises #1-9, the sample space is {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.
1. This is an "and" statement, so we need both an H and a 3. P(H3) = 1/12.

2. This is an "and" statement, so we need both a T and a 7, which isn't possible. P(T and 7) = 0.

3. This is an "and" statement, so we need an H and an odd number. P(H and Odd) = 3/12.

4. This is an "and" statement. There are no numbers greater then 6, so P(H and #>6) = 0/12 = 0.

5. This is an "or" statement. We cannot get a 7, but half the outcomes have tails. P(T or 7) = 6/12.

6. This is an "or" statement. Since all the numbers are less than 9, P(T or #<9) = 12/12 = 1.

7. Odds against statements are the ratio of fails to successes. Since there are six outcomes without a H, there are six fails. Likewise, there are six successes. 6:6.

8. Of the twelve possible outcomes, only H6 and T6 have 6s. Those are the successes. That makes the odds against a 6 are 10:2.

9. H2, H4, H6, T2, T4, T6 are all successes, while the other six outcomes are fails. The odds against an even number are 6:6.

For Exercises 10-14, there are 36 outcomes.
10. There are five ways to get a 6: (1, 5), (2, 4), (3, 3), (4, 2), and (5, 1). P(6) = 5/36

11. There are two ways to get a 3, four ways to get a 5, six ways to get a 7, four ways to get a 9, and 2 ways to get an 11. P(odd) = 18/36.

12. The multiples of 3 are 3, 6, 9 and 12. There are two ways to get a 3, five ways for a 6, four ways to get a 9, and one way for a 12. P(multiple of 3) = 12/36.

13. There are 2 ways to get an 11, making (5, 6) and (6, 5) the successes. The other 34 rolls are fails. Thus, the odds against an 11 are 34:2 = 17:1.

14. There are four ways to get a 5. That makes 32 fails. So, the odds against a 5 are 32:4 = 8:1.

For Exercises #15-19, there are 24 jellybeans in the bag.
15. P(orange) = 7/24

16. P(green OR yellow) = 6/24

17. P(neither red nor purple) = 16/24

18. P(not yellow) = 22/24

19. There are no black jellybeans in the bag. P(black) = 0.

20. There is only one queen of hearts. P(queen of hearts) = 1/52

21. There are four 6s. P(6) = 4/52

22. Half the cards are black. P(black) = 26/52

23. There are two red 9s. P(red 9) = 2/52

24. There are 13 hearts and three more cards that are aces (don’t count the aces of hearts twice!). P(ace OR heart) = 16/52

25. There is only one card that is an ace AND a heart. P(ace AND heart) = 1/52

26. There are three face cards in each suit. P(face card) = 12/52

27. There are four kings and 48 cards that are not kings. Odds against a king = 48:4

28. There are 16 cards that are either a diamond or a queen (don’t count the queen of diamonds twice), which means there are 36 other cards. So, the odds against a queen or a diamond are 36:16.

29. Since the odds against the event are 5:2, there are 5 fails and 2 successes, which makes a total of 7 outcomes. That makes the P(success) = 2/7.

30. Since the odds against the event are 3:7, there are 3 fails and 7 successes, which makes a total of 10 outcomes. That makes the P(success) = 7/10.

4.3: EXPECTED VALUE & FAIR PRICE

In the Long Run...

Remember that phrase: In the long run... Although probabilities are used to make a prediction as to what should happen for one particular event, the expected value calculations are all based on the long-term expectations of an event. While we expect 1 out of every 2 flips of a coin to be heads, sometimes we might get 5 tails in a row, so there are no guarantees in the short term. What a probability value tells us is that while a few flips might give strange results, if we flip that coin 1000 times, it is very likely that the number of heads we end up with will be fairly close to 500. This idea is known as **The Law of Large Numbers**.

Expected Value

The **expected value** (EV) calculation is used to determine the value of a business venture or a game over the long run. Sometimes, instead of saying "expected value," we may simply say "expectation."

Expected value is not the value of a particular event.
It is the long run average value if the event was repeated many times.

Example 1: Assume we have a game in which we either win \$40 or lose \$1, and we play the game 20 times - winning twice and losing the other 18 times. How much money will we have at the end of those 20 turns?

2 wins × \$40 + 18 losses × (-\$1) = \$80 + (-\$18) = \$62

Observe that we are more likely to lose than we are to win. But when we win, the amount won makes up for the frequent losses.

Since we played 20 times, the average amount we won per turn was \$62/20 = \$3.10. In other words, the long run results would have been the same as if we had won \$3.10 each time we played. \$3.10 is the expected value of each turn for that game.

To calculate the expected value of an event, multiply the probability of each outcome in the event by the value of that outcome, and then add those results together.

EV = (Prob. of Outcome #1) × (Value of #1) + (Prob. of Outcome #2) × (Value of #2)

Let's examine a situation similar to the game in Example 1.

Example 2: A game has 18 white balls and 2 red balls in a bag, and you will randomly select one ball from the bag. If the selected ball is white, you lose \$1. And, if the ball is red, you win \$40. What is the expected value of the game?

P(White) = 18/20, and the value of selecting a white ball is -\$1.
P(Red) = 2/20, and the value of selecting a red ball is \$40.

EV = P(White) × (White Ball Value) + P(Red) × (Red Ball Value)
EV = (18/20)×(-\$1) + (2/20)×(\$40) = -\$0.90 + \$4.00 = \$3.10

Let's find the Expected Value of that same game, with one small change.

Example 3: A game has 18 white balls and 2 red balls in a bag, and you will randomly select one ball from the bag. If the selected ball is white, you lose \$1. And, if the ball is red, you win \$10. What is the expected value of the game?

P(White) = 18/20, and the value of selecting a white ball is -\$1.
P(Red) = 2/20, and the value of selecting a red ball is \$10.

EV = P(White) × (White Ball Value) + P(Red) × (Red Ball Value)
EV = (18/20)×(-\$1) + (2/20)×(\$10) = -\$0.90 + \$1.00 = \$0.10

What if we win only \$5?

Example 4: A game has 18 white balls and 2 red balls in a bag, and you will randomly select one ball from the bag. If the selected ball is white, you lose \$1. And, if the ball is red, you win \$5. What is the expected value of the game?

P(White) = 18/20, and the value of selecting a white ball is -\$1.
P(Red) = 2/20, and the value of selecting a red ball is \$5.

EV = P(White) × (White Ball Value) + P(Red) × (Red Ball Value)
EV = (18/20)×(-\$1) + (2/20)(\$5) = -\$0.90 + \$0.50 = -\$0.40

Notice the version with a \$5 prize has a negative expected value. That means we would expect to lose an average of 40¢ per play. If we were to play this game 20 times, we would expect to lose a total of \$8.

How about a game with more than two possible outcomes?

Example 5: A game has one yellow ball, two blue balls, two red balls, and three green balls in a bag, and you will randomly select one ball from the bag. If the selected ball is yellow, you lose \$2. If the ball is blue, you lose \$1. If the ball is red, you win \$8. If the ball is green, you lose \$4. What is the expected value of playing the game?

First, find the probability and value of each outcome. Remember to list losing values as negative amounts.

Yellow Ball: P(Yellow) = 1/8, Value = -\$2.
Blue Ball: P(Blue) = 2/8, Value = -\$1.
Red Ball: P(Red) = 2/8, Value = \$8.
Green Ball: P(Green) = 3/8, Value = -\$4.

EV = (1/8)(-\$2) + (2/8)(-\$1) + (2/8)(\$8) + (3/8)(-\$4) = \$0/8 = \$0

Raffles

Raffles are a game in which the player must pay up-front for the opportunity to take part. The money paid for a raffle ticket is not added on to the prize money, and that has to be considered when calculating the expected value of a raffle. For example, if a player pays \$5 for a raffle ticket and the prize is \$100, the actual winnings would be \$95, not \$100.

Example 6: A school club is fundraising with a raffle. If they sell 100 tickets for \$5 each, and give a \$100 prize to a single winner, what is the expected value of a single ticket in for the raffle?

For a single ticket, there are exactly two outcomes: win or lose. So, start by finding the probability and value for each outcome. Keep in mind, if we have the winning ticket, we get \$100, but we had to pay \$5 to get the ticket in the first place.

Win: P(Win) = 1/100, Value = \$100 - \$5 = \$95
Lose: P(Lose) = 99/100, Value = \$0 - \$5 = -\$5

EV = (1/100)(\$95) + (99/100)(-\$5) = (\$95 - \$495)/100 = -\$400/100 = -\$4

Games with positive expected values are good to play. If we play them a lot, sometimes we will win, and sometimes we will lose. On average, however, we can expect to make money in the long run.

Inversely, if a game has a negative expectation, the long run average would see players losing money. We still may win some money in a few of the attempts, but the long run average would be negative. Do realize, that does not mean we should avoid games with negative expected values.

A negative EV for the player means a positive EV for the group sponsoring the game. State-run lotteries have a negative EV for the players, which means they make money for the State. That money often gets invested into things like infrastructure and education.

In the previous example, the EV for the player was -$4. Assuming the club sold all the tickets, they made $500. Then they gave $100 to one lucky winner. The net result is the club making a profit of $400.

Example 7: One thousand raffle tickets are sold for $1 each. One first prize of $500, and two second prizes of $100 will be awarded. What is the expected value of a single ticket?

For a single ticket, there are three possible outcomes: win the first prize, win one of the second prizes, or lose. Find the probability and value for each outcome, keeping in mind the ticket cost $1.

First Prize: P(First Prize) = 1/1000, Value = $500 - $1 = $499
Second Prize: P(Second Prize) = 2/1000, Value = $100 - $1 = $99
Lose: P(Lose) = 997/1000, Value = $0 - $1 = -$1

EV = (1/1000)($499) + (2/1000)($99) + (997/1000)(-$1) = ($499 + $198 - $997)/1000 = -$0.30

Fair Games

A game is considered to be "fair" - in the mathematical sense - when the expected value is equal to $0, meaning we should break even over the long run, and there is no advantage for the player or sponsor. The game in Example 5 is an example of a **fair game**.

Fair Price to Play

The **fair price to pay** (FP) for playing a game is the amount that would make the EV zero. To find the FP, we simply find the EV without making any deduction for the cost to play.

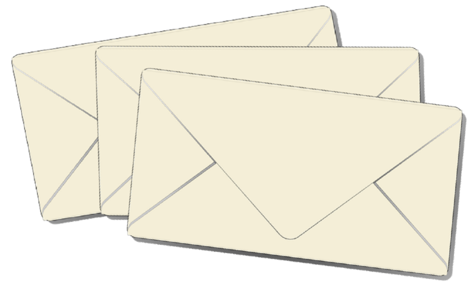
The fair price for a game is the cost that would make the expected value $0.

Example 8: A high school is selling 500 raffle tickets for a free dinner for two, valued at $75. What would be a fair price to pay for one ticket?

EV = (1/500)×($75) = $0.15, so the fair price to pay for a ticket would be 15¢. Naturally, the school would not sell them for 15¢ each, but likely for $1 each, so that they can make a profit on the raffle.

Finally, let's consider one more game.

Example 9: A carnival game has a basket with three envelopes in it, of which a player will pick one and keep the contents. One envelope has a \$1 bill, one has a \$10 bill in it, and the third envelope has a \$100 bill in it. What is the fair price to play for this game? Would you pay \$20 to play this game?



EV = (1/3)(\$1) + (1/3)(\$10) + (1/3)(\$100) = \$111/3 = \$37

Thus, if we paid \$37 to play this game, the EV would be \$0. That makes the FP = \$37.

If the cost to play is anything less than \$37, the long ran average would be positive. So, even though you may be reluctant, forking over \$20 to play is actually a good move.

Section 4.3 Exercises

1. A game has 19 white balls and 1 red ball in a bag, and you will randomly select one ball from the bag. If the selected ball is white, you lose \$1. And, if the ball is red, you win \$100. What is the expected value of the game?

2. A game has 24 white balls and 16 red balls in a bag, and you will randomly select one ball from the bag. If the selected ball is white, you lose \$1. And, if the ball is red, you win \$10. What is the expected value of the game?

3. There are ten envelopes on a table and you get to pick one at random. In one of the envelopes there is a \$100 bill, and in the other nine there are blank pieces of paper. What is the expected value for picking an envelope?

4. There are ten envelopes on a table and you get to pick one at random. In one of the envelopes there is a \$100 bill, two of them have a \$10 bill, and in the other seven there are blank pieces of paper. What is the expected value for picking an envelope?

5. 100 tickets are sold at a cost of \$5 each. The prize for the winner is \$200. What is the expected value of a ticket?

6. 1000 tickets are sold at a cost of \$2 each. The prize for the winner is \$500. What is the expected value of a ticket?

7. 500 tickets will be sold for a raffle at a cost of $2 each. The prize for the winner is $750. What is the expected value of raffle ticket?

8. There are 5 black balls, 4 red balls, and 11 white balls in a jar, and you get to randomly pick one ball without looking. If you draw a black ball, you win $5. If you pick a red one, you lose $1. If you pick a white one, you lose $4. What is the expected value of picking a ball?

9. A bag contains 1 black ball, 4 red balls, 3 green balls, and 2 white ones, and you get to randomly pick one ball without looking. If you pick the black one, you win $7. If you pick a red one, you lose $3. If you pick a green ball, you lose $2. If you pick a white ball, you win $4. What is the expected value of picking a ball?

10. Four baseball caps are on a table. One cap has a $1 bill under it, another has a $5 bill, another has a $50 bill, and the last one has a $100 bill. You can select one cap and keep the amount of money that is underneath. What is the expected value for playing this game?

11. A small organization is renting an outdoor arena for $2000 and hosting a concert. If is it sunny outside, they can expect to earn $5000 in ticket sales. If it rains, the company will lose the $2000 fee and the event will be cancelled. If there is a 20% chance of rain on the day of the event, what is the expected value of the concert?

12. A raffle is going to be held in which the winner will get $500. If 1000 tickets are to be sold, what is the fair price for a ticket?

13. A raffle is going to be held in which the winner will get \$1500. If 200 tickets are to be sold, what is the fair price for a ticket?

14. A raffle is going to be held in which there is a First Prize worth \$300, and a Second Prize worth \$100. If 100 tickets are to be sold, what is the fair price for a ticket?

15. A raffle is going to be held in which there is a First Prize worth \$300, a Second Prize worth \$100, and two Third Prizes that are worth \$50. If 100 tickets are to be sold, what is the fair price for a ticket?

16. A school carnival involves a game in which the player rolls a single 6-sided die. If the result is an odd number, the player wins \$1. If the result is an even number, the player wins the amount of money shown on the die. That is, a roll of 2 means the player wins \$2, a roll of 4 means the player wins \$4, a roll of 6 means the player wins \$6. What is a fair price to pay for playing this game? If the school would like to profit from this game, what are they likely to charge?

Section 4.3 Exercise Solutions

1. EV = (1/20)($100) + (19/20)(-$1) = $81/20 = $4.05

2. EV = (16/40)($10) + (24/40)(-$1) = $136/40 = $3.40

3. EV = (1/10)($100) + (9/10)($0) = $100/10 = $10

4. EV = (1/10)($100) + (2/10)($10) + (7/10)($0) = $120/10 = $12

5. EV = (1/100)($195) + (99/100)(-$5) = -$300/100 = -$3

6. EV = (1/1000)($498) + (999/1000)(-$2) = -$1500/1000 = -$1.50

7. EV = (1/500)($748) + (499/500)(-$2) = -$250/500 = -$0.50

8. EV = (5/20)($5) + (4/20)(-$1) + (11/20)(-$4) = -$23/20 = -$1.15

9. EV = (1/10)($7) + (4/10)(-$3) + (3/10)(-$2) + (2/10)($4) = -$3/10 = -$0.30

10. EV = (1/4)($1) + (1/4)($5) + (1/4)($50) + 1/4)($100) = $156/4 = $39

11. EV = P(Sun)×Profit + P(Rain)×Loss = 0.8($5000 - $2000) + 0.2(-$2000) = $2000

12. FP = (1/1000)($500) = $0.50

13. FP = (1/200)($1500) = $7.50

14. FP = (1/100)($300) + (1/100)($100) = $400/100 = $4

15. FP = (1/100)($300) + (1/100)($100) + (2/100)($50) = $500/100 = $5

16. FP = P(Odd)×$Won + P(2)×$Won + P(4)×$Won + P(6)×$Won
    FP = (3/6)($1) + (1/6)($2) + (1/6)($4) + (1/6)($6) = $15/6 = $2.50
    The school will profit if the cost to play the game is anything more than $2.50. However, if they charge too much, no one will play the game. Charging $3 is reasonable.

4.4: AVERAGES

Mean

The thing everyone loves to call the "average" is actually called the **mean**. This is what we get when we "add them all up and divide by how many there are." The term average is a general term that actually applies to one of several different measures of **central tendency**.

Example 1: Find the mean of the set {4, 6, 9, 3, 2}.

Mean = (4 + 6 + 9 + 3 + 2)/5 = 24/5 = 4.8

Sometimes we may need to use our algebra skills to find a missing value that would produce a given mean.

Example 2: A student's test scores on the first three tests were 72, 78, and 89. What score would be needed on the fourth test to achieve a mean score of 80?

We want the mean of four scores to be 80. In other words, we want the sum of the four scores, divided by 4 to equal 80. If we let x stand for the fourth test score, we have:

(72 + 78 + 89 + x)/4 = 80
72 + 78 + 89 + x = 320
239 + x = 320
x = 320 – 239 = 81

The student needs a score of 81 on the fourth test to achieve a mean of 80 for the four tests.

Median

A highway median is the part in the middle. Similarly, the **median** of a set of data is the middle number in the ordered set.

Be careful. The data values must be arranged in order before we can find the median. This order can be from lowest to highest, or from highest to lowest, but either way, the median is the value in the middle. For example, if our data is {3, 3, 8, 7, 5}, we might be confused into thinking the median is 8 because it appears in the middle. But remember, the numbers must be ordered first. So, since our ordered data is actually {3, 3, 5, 7, 8} when arranged from lowest to highest, we state the median is 5.

The nice thing about finding the middle is knowing the data set is split in two equal parts. Half the values are above the median and have the values are below it.

COMMON MISTAKE

Be sure to arrange the numbers in order BEFORE you look for the median.

Example 3: Find the median of the set {4, 6, 9, 3, 2}.

Arranging the numbers in order, the set is {2, 3, 4, 6, 9}. The number in the middle is the 4.
Median = 4

What happens to the median if there is an even number of data values? The median is then found by finding the mean of the two numbers in the middle. For example, if the data is {2, 3, 5, 7, 8, 9}, there is not a single middle number. Since 5 and 7 are the two numbers in the middle, the median would be the mean of those two numbers, or (5 + 7)/2 = 6. Thus, the median for that data would be 6.

Example 4: Find the median of the set {5, 7, 3, 13, 6, 6, 17, 12, 10, 9}.

Arranged from least to greatest, the set is {3, 5, 6, 6, 7, 9, 10, 12, 12, 17}. Of the ten values, the 7 and the 9 are in the middle. Thus, the median = (7 + 9)/2 = 8.

Depending on the situation, since half data values in the set are above the median and half are below it, the median can often be a better descriptor of the central tendency of the set than the mean. For example, medians are frequently used when describing incomes or home prices for a specific area. Whereas the mean home value in a town may get skewed by a few overly expensive homes, the median home value is always the one in the middle, regardless of the specific home values. Doubling the value of the most expensive home could significantly alter the mean, but it would not change the middle of the set, at all.

Example 5: A small tech company has four employees that make \$50,000 per year and a CEO that makes \$200,000 per year. What is the mean salary at the company? What is the median salary? Which average is a better description of the "average employee salary?"

The mean salary is (4×\$50,000 + \$200,000/5 = \$400,000/5 = \$80,000
The median of {\$50,000, \$50,000, \$50,000, \$50,000, \$200,000) is \$50,000.

"Average" can be either the mean or the median. If the company was trying to impress someone, it is perfectly accurate to say, "The average salary is \$80,000 per year." That does, however, seem a bit disingenuous. In this case, it seems a bit more honest to use the median.

Mode

A third descriptive measure of a data is the mode. The **mode** is the single data value that occurs most often within the data set. In the previous example, the modal salary of the company is $50,000.

Example 6: Find the mean, median and mode of the exam scores listed below. Round your answers to the nearest tenth, as necessary.

93, 97, 59, 71, 57, 84, 89, 79, 79, 88, 68, 91, 76, 87, 94, 73, 58, 85, 82, 38

For the mean, the sum of the numbers is 1548, and 1548/20 = 77.4.

For the median, first arrange the numbers from lowest to highest. Since there is even number of values, we must find the mean of the two in the middle. (79+82)/2 = 80.5.

The mode is 79, as it is the most common value.

If there are two data values that occur more often than the others, then we refer to the data as **bimodal**. For example, the data set {2, 3, 4, 5, 7, 2, 7, 8, 2, 7, 7, 2} is bimodal, as the 2 and the 7 occur four times each. If there is not a single value (or a pair of values) that occurs most often, we say there is no mode. Even though modal values don't have to appear around the middle of a data (that is, they are not considered a measure of "central tendency"), they can often be useful in describing the overall appearance of the group.

Grade Point Average

Students need to be able to compute their own **Grade Point Average**, also known as the **GPA**. The GPA is a weighted average. That is, each grade has a weight associated with it that corresponds to the number of credits of the class in which the grade was earned. An A in a 4-credit class carries more weight than an A in a 3-credit class. Before we get to the calculation, we need to know the values of specific grades.

Grade	Value	Grade	Value	Grade	Value	Grade	Value	Grade	Value
A	4.0	B+	3.3	C+	2.3	D+	1.3	F	0.0
A-	3.7	B	3.0	C	2.0	D	1.0		
		B-	2.7	C-	1.7	D-	0.7		

Classes grades of I (Incomplete), P (Pass), S (Satisfactory), U (Unsatisfactory), W (Withdrawal), NR (Not Reported), or AU (Audit) are not included in the GPA calculation.

Take a good look at the grade values. We should not need to memorize the values if we recognize the values of A, B, C, D, and F are 4, 3, 2, 1, and 0, respectively. Then notice that "plus" grades are 0.3 points higher, and "minus" grades are lowered by 0.3. Many high schools offer accelerated classes that would allow students to earn a grade of A+, which would carry a weight of 4.3. Colleges and universities, however, rarely do so. Additionally, we do not have grades of F+ and F-.

To compute a GPA, we first multiply the value of the class grade by the corresponding weight, which is the number of credits for the class. That product gives us the number of Grade Points for the class. Then, we add together the Grade Points for all the classes and divide by the total number of credits taken. Even though individual class grade values are listed to the tenth of a point, schools will publish GPA's that are rounded to the nearest hundredth or thousandth of a point. In our calculations, we need to be sure to follow any listed rounding instructions.

Example 7: Last semester, Mary earned an A in her 4-credit ENG 101 class, a B in her 3-credit COM 117 class, and a C+ in a 3-credit MATH 112 class. What was her GPA for the semester? Round your answer to the nearest hundredth.

GPA = (4×4.0 + 3×3.0 + 3×2.3)/10 = 31.9/10 = 3.19

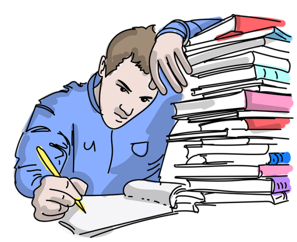
When finding GPAs, we need to be sure to weight the grades by the number of credits. A common mistake would be to average the grades, instead of the grade points. That is, in the above example, some students ignore the credits and attempt to find the average grade by dividing the sum of the grade values by the number of classes. (4.0 + 3.0 + 2.3)/3 = 9.3/3 = 3.1 is incorrect. It's close, but still incorrect. As a reminder, we need to divide by the number of credits, not the number of classes.

COMMON MISTAKE

A common mistake is finding the mean grade value,
without weighting the grades by the number of credits.

Remember, we need to divide by the number of credits, not the number of classes.

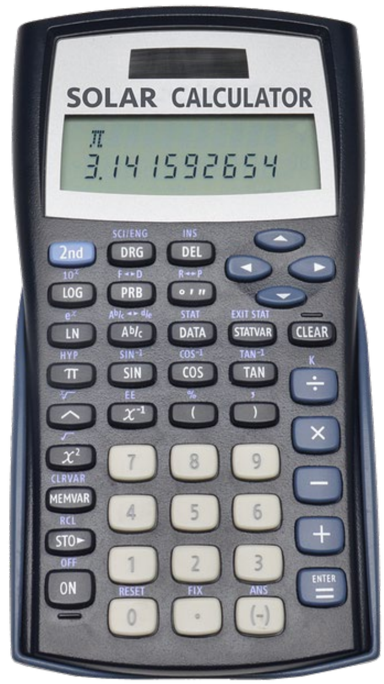
Example 8: Last semester, John earned an A- in his 3-credit ENG 103 class, a B in his 4-credit HIST 107 class, a C+ in a 3-credit MATH 112 class, and an A in a 2-credit IS 241 class. What was his GPA for the semester? Round your answer to the nearest thousandth.



GPA = (3×3.7 + 4×3.0 + 3×2.3 + 2×4.0)/12 = 38.0/12 = 3.16666… → 3.167

Using Your Calculator

In the previous example, the final GPA calculation looked like (3×3.7 + 4×3.0 + 3×2.3 + 2×4.0)/12. A critical part of that line is the parentheses. Without them, the result would be different – and incorrect.



Remember, scientific calculators obey the order of operations. Without the parentheses, the calculation would divide only the adjacent number, instead of the entire sum.

$$9 + 6/3 = 9 + 2 = 11$$
$$(9 + 6)/3 = 15/3 = 5$$

That doesn't mean we always need to use the parentheses keys on a calculator. If we don't want to use those keys (or forget to hit the open parenthesis key), we can simply hit the equals key just before we do the division.

Section 4.4 Exercises

For Exercises #1-4, find the mean, median and mode of each data set. Round your answers to the nearest tenth, as needed.

1. {1, 3, 5, 5, 7, 8, 9, 11}

2. {5, 13, 8, 4, 7, 2, 11, 15, 3}

3. {37, 52, 84, 99, 73, 17}

4. {4, 16, 9, 4, 2, 4, 1}

5. During the month of November 2008, the Oakland Raiders played five football games. In those games, the team scored 0, 6, 15, 31, and 13 points, respectively. Find the mean, median and mode for this data set. Round your answers to the nearest tenth, as needed. Did the Raiders win all five games?

6. The mean score on a set of 15 exams is 74. What is the sum of the 15 exam scores?

7. A class of 12 students has taken an exam, and the mean of their scores is 71. One student takes the exam late, and scores 92. After including the new score, what is the mean score for all 13 exams? If you get a decimal answer, you should round to the nearest hundredth.

8. The twenty students in Mr. Edmondson's class earned a mean score of 76 on an exam. Taking the same exam, the ten students in Mrs. Wilkinson's class earned a mean score of 86. What is the mean when these teachers combine the scores of their students? If you get a decimal answer, you should round to the nearest hundredth.

9. After six exams, Carl has a mean score of 78.5. With only one exam remaining in the class, what is the minimum score Carl will need on that exam to have an overall mean of 80?

10. Create a set of seven data values in which the mean is higher than the median.

11. Can the mean be a negative number? Explain your answer and give an example.

12. This semester, Amy earned an A in her 5-credit chemistry class, a C in her 1-credit art class, and a B in her 3-credit Spanish class. What is her GPA for the semester? Round your answer to the nearest thousandth.

13. This semester, Peter earned a C+ in his 3-credit math class, a B in his 3-credit English class, a C in his 4-credit history class, and an A in a 1-credit P.E. class. What is his GPA for the semester? Round your answer to the nearest hundredth.

14. This semester, Joy earned an A- in her 4-credit chemistry class, a D in her 2-credit art class, and an F in her 3-credit Spanish class. What is her GPA for the semester? Round your answer to the nearest thousandth.

Section 4.4 Exercise Solutions

1. For the mean: Mean = 49/8 = 6.125 → 6.1
   For the median, first notice the data is already ordered. Median = (5 + 7)/2 = 6
   Mode = 5

2. For the mean: Mean = 68/9 = 7/555… → 7.6
   For the median, order the data: {2, 3, 4, 5, 7, 8, 11, 13, 15}. Median = 7
   No Mode

3. For the mean: Mean = 362/6 = 60.333… → 60.3
   For the median, order the data: {17, 37, 52, 73, 84, 99}. Median = (52 + 73)/2 = 62.5
   No Mode

4. For the mean: Mean = 40/7 = 5.714285… → 5.7
   For the median, order the data: {1, 2, 4, 4, 4, 9, 16}. Median = 4
   Mode = 4

5. For the mean: Mean = 65/5 =
   For the median, order the data: {0, 6, 13, 15, 31}. Median = 13
   No Mode
   Since they could not win a game in which they scored zero points, they did not win all five games.

6. With 15 scores, the mean = sum/15. Algebra tells us we can multiply both sides of the equation by 15 to get 15(mean) = sum. 15(74) = 1110

7. With 12 students in the books, the sum of those 12 scores is 12(71) = 852. Add in the new score of 92 to get 944. Then divide the new sum by 13. 944/13 = 72.61538… → 72.62

8. The sum of the scores for Mr. Edmondson's twenty students is 20(76) = 1520. Likewise, the sum of the scores for Mrs. Wilkinson's ten students is 10(86) = 860. Adding those individual sums together and dividing by a total of 30 students gives an overall mean of (1520 + 86)/30 = 79.333… → 79.33.

9. After 6 exams, the sum of his scores is 6(78.5) = 471. The new sum will be 471 + x, where x is the score on the seventh test. Thus, the new mean would be (471 + x)/7. Set that equal to 80, and solve for x. (471 + x)/7 = 80 → 471 + x = 560 → x = 560 – 471 = 89

10. Answers may vary. Find the mean and median to check and see if your data works. One possible answer would be {1, 1, 1, 1, 1, 1, 8}. The median is 1, while the mean is 2.

11. Yes. An example would be the mean low temperature in Anchorage, Alaska during the month of January. The mean of a bunch of negative numbers would definitely be negative.

12. GPA = (5×4.0 + 1×2.0 + 3×3.0)/9 = 31/9 = 3.4444…. → 3.444

13. GPA = (3×2.3 + 3×3.0 + 4×2.0 + 1×4.0)/11 = 27.9/11 = 2.53636…. → 2.54

14. GPA = (4×3.7 + 2×1.0 + 3×0.0)/9 = 16.8/9 = 1.8666…. → 1.867

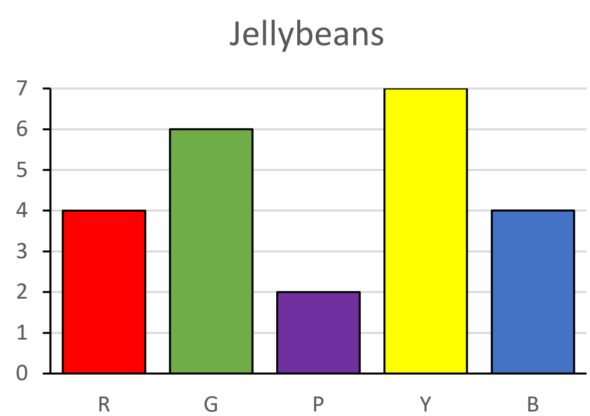
4.5: CHARTS & GRAPHS

Bar Graphs

A **bar graph**, also called a **bar chart**, is a visual way to present the differences between the distinct quantities of data. Each of the different things being counted are sorted into categories called **bins**. and are typically listed across a horizontal axis. Then, the number of items in each category is indicated by the height of the corresponding bars. A bar graph allows the reader to focus on the height of the bars instead of the number of each item.

A bar graph has clear spaces between the bars to indicate each category is separate and distinct, and can appear in any order. The bars on a bar graph can be horizontal, but we will stick with a vertical representation.

Example 1: A bag contains 4 red jellybeans (R), 6 green jellybeans (G), 2 purple jellybeans (P), 7 yellow jellybeans (Y), and 4 blue jellybeans (B). Create a bar graph that displays this data.



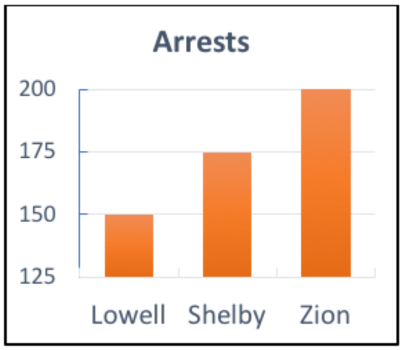
Making a bar graph is pretty straight-forward; where people run into problems is general sloppiness. When making a bar graph - or any graph, for that much - we need to be very neat and consistent. Leaving large gaps between the bars makes the graph look awkward and/or using inconsistent spacing of the quantities on the vertical axis can deceive the viewer.

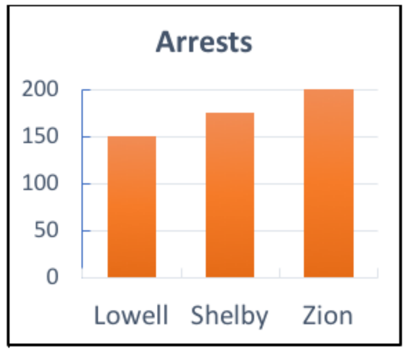
COMMON MISTAKE

The most common mistakes in making charts and graphs are a direct result of neglecting to be neat, consistent, and organized.

Unfortunately, an individual or group wanting to intentionally deceive a viewer may use choose to display a bar graph in a biased fashion. With appropriate labels, biased bar charts are not incorrect. However, by drawing the viewer's attention to the bars, the data become very easy to misinterpret.

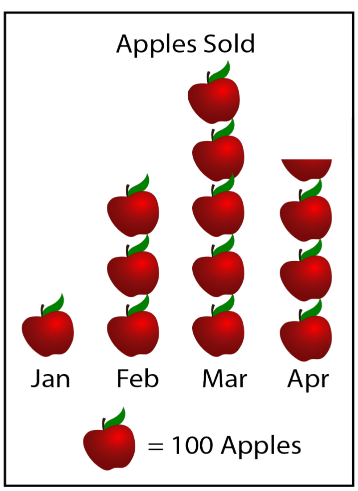
Consider the following bar graphs, which display the differences between arrests made last year in three different small towns. In a presentation, the town of Lowell is attempting convince viewers they are much more likely to get arrested in neighboring towns, as the graph on the left displays bars for the other cities that are considerable larger than the one for Lowell. Even though both graphs are technically correct, we can see that the differences between the bars is much less dramatic when the vertical axis starts at 0 instead of 125.





Bar graphs are one of the most common ways to visually represent groups of data. Sometime the graphs can be made more appealing using stacked or differently-sized graphics in place of simple colored bars. This type of bar graph is called a **pictograph**. When a pictograph is used, a key must be provided to indicate the value of the symbol.

Example 2: Bob's orchard sold 100 apples in January, 300 in February, 500 in March, and 350 apples in April. Using apples for bars, create a pictograph of this data.



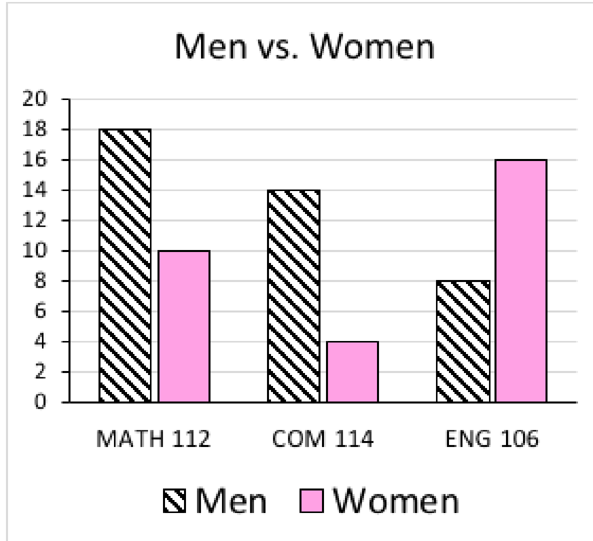
If we want to compare multiple sets of similar data, we can lump multiple bar graph into the same display. **Clustered bar graphs** contain a lot of information and, hence, can look a little busy. The information, however, is pretty straight-forward.

Example 3: Given the clustered bar graph, answer the following questions.

a. What class has more women than men?

b. How many students are in MATH 112?

c. How many more men than women are there in COM 114?



a. ENG 106 has more women than men.

b. MATH 112 has 18 men and 10 women, for a total of 28 students.

c. There are 14 men and 4 women, so there are 10 more men than women in COM 114

Line Graphs

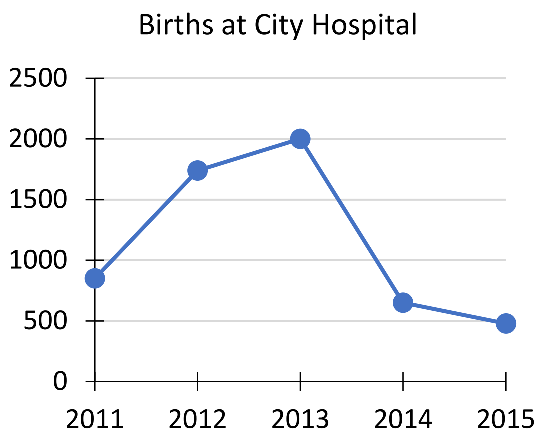
Line graphs are another way to represent data, especially when that data is collected over time. Whereas bar graphs direct the viewer's attention to the distinct data amounts, line graphs draw the viewer's attention to the increase or decrease in the amount of an item over a specified time.

The horizontal axis of a line graph represents a timeline, while values of the data are indicated along the vertical axis. When constructing a line graph, we need to be sure to put a clear marker at the point representing the amount at a specific time, and then draw straight line segments between consecutive markers.

Like with bar graphs, the most common problems deal with neatness – or lack thereof. Since the focus of a line graph is on the line segments between the markers, we need to make sure they are straight. If we do not have ruler handy, many things (cell phones, business cards, book covers, etc.) can be substituted for a straightedge. Alternatively, if we create the line graph using computer software, neatness usually is not a problem.

Similar to bar graphs, multiple sets of data can be displayed in a single **clustered line graph**. In these, we need to be sure to provide a clear distinction between the data sets. This is usually done with different colors, labels and even marker types (e.g. use circles for one set and squares for a second set of data).

Example 4: In 2011 there were 850 babies born at City Hospital. In 2012 there were 1740 births, and in 2013 there were 2001 births. In 2014 a new hospital opened on the other side of town and the number of births at City Hospital dropped to 650. Then, in 2015 there were 478 babies born at City Hospital. Create a line graph that represents this data.



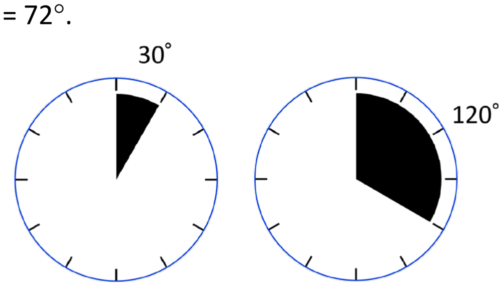
In the previous example, a presenter can draw attention to the dramatic decrease in births between 2013 and 2014. Then a discussion can ensue about the reasons behind the decrease, along with potential plans of action.

Pie Charts

Pie charts, also called **circle graphs**, are used to show percentage relationships between disjoint data. That is, the data cannot overlap, and must describe all of the alternatives in a set. If you and some friends are splitting a circular pie, the size of each slice (also called a **sector**,) corresponds to the percent or fraction of the pie each person gets.

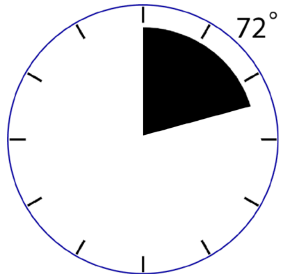
The most important number to have when creating a pie chart is the total number of data values. When making a pie chart, we need to decide how many degrees of the circle should be given to each category of data. Remember, a circle has 360 degrees in total, so that number will also be very important to us. So, how do we divide up those 360°?

The degrees dedicated to each sector in the circle graph can be found by multiplying the corresponding fraction by 360°. For example, if we have a class of 20 students with 4 of those students earning As, the number of degrees dedicated to the A sector will be (4/20)(360˚) = 72°.



The next step is figuring out how to draw a 72°-degree sector. To do this, picture an analog clock. Every hour on the clock constitutes 1/12 of a circle, which corresponds to (1/12)(360°) = 30°. So, if we open up a sector from 12 o'clock to 1 o'clock, it would be 30°. Likewise, a sector from 12 o'clock to 4 o'clock would be (4/12)(360°) = 120°.

What about 72°? Since the sector from 12 to 2 would be 60° and the sector from 12 to 3 would be 90° we can see that 72°, is equal to a sector from 12 o'clock to a little less than halfway between 2 and 3 o'clock.



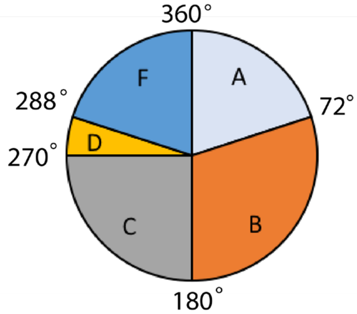
Once we understand how to make the appropriate sector sizes, the last piece of the puzzle is knowing where to start and stop them. The initial sector should always start at 12 o'clock. Then, the next sector will start where the first sector ended.

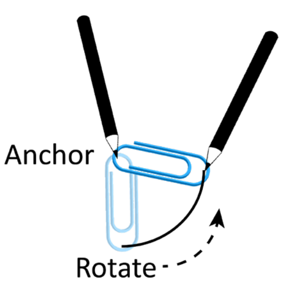
For example, if the first sector is 72° and the next sector is 108°, that second sector will start at 72° and end at 180˚ (72° + 108° = 180°). Then continue this process, starting each new sector where the previous sector ended, and moving in a clockwise direction. Keep in mind, 100% of the data should account for the entire 360° of the circle.

Although we could, technically, start the first sector at any angle, and even arrange them in any order (as long as they are the correct size), for consistency, we will start the first sector at 12 o'clock, and proceed clockwise, keeping the sectors in the same order in which they are presented.

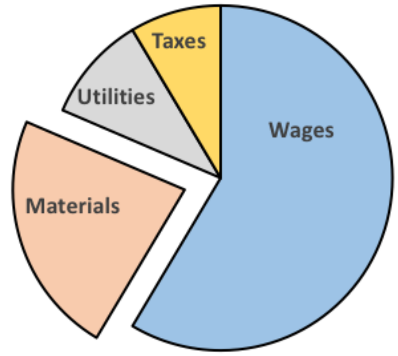
Example 5: In a class with 20 students, 4 have earned an A. 6 students get Bs, 5 earn Cs, there is 1 D and 4 Fs. Make a pie chart that represents the class grades.

The A sector is (4/20)(360°) = 72°.
The B sector is (6/20)(360°) = 108°, and ranges from 72° to 180°.
The C sector is (5/20)(360°) = 90°, and ranges from 180° to 270°.
The D sector is (1/20)(360°) = 18°, and ranges from 270° to 288°.
The F sector is (4/20)(360°) = 72°, and completes the circle at 360°.





Drawing circles can be a bit challenging, especially if we don't happen to have a compass handy. We can get circles by tracing circular objects, such as bottle caps, but it may be tough to find an object that is the desired size. If, however, we have a paperclip and a couple pencils, we can get a perfect circle by using a pencil to anchor one end of the paperclip and then use the other pencil to trace out a circle using the other end of the paperclip. Basically, the length of the paperclip is the radius of our circle. Then, for a differently-sized circle, we can simply bend or unfold the paperclip.



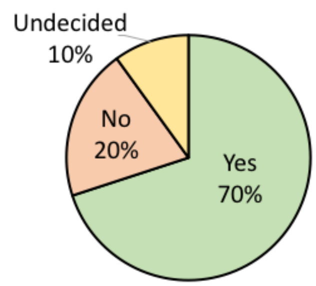
Sometimes a presenter may want to draw attention to a specific sector in a pie chart. To do this, we can use our generating software to drag the desired sector away from the center of the circle. This type of chart is called an **exploded pie chart**.

For example, when talking about how to reduce costs, an employee may want to discuss the costs associated with the materials that get purchased. The specific data used to generate the sector size, however, remains unchanged.

Generating Data from a Pie Chart

Reading bar graphs and line graphs just boils down to looking at the height of the bars/markers. Pie charts, however, as not as straight forward. If we are given a pie chart with the size of each sector and the total amount of data in the given set, we can work backwards to generate the data set. For example, if a sector is 90°, we know that data represents 90°/360° = 25% of the entire data. Thus, if we are told there were 3000 people surveyed, we can tell that the corresponding sector represents (0.25)(3000) = 750 people.

Example 6: The given pie chart represents the preferences of 5000 voters about whether or not they supported Ballot Initiative #1. How many of the voters support the initiative? How many voters are undecided?



Support: 70% of 5000 = (0.70)(5000) = 3500 Voters
Undecided: 10% of 5000 = (0.10)(5000) = 500 Voters

Which Type of Graph is Best?

There are many, many, many more types of charts and graphs that can be used to represent a set of data. If, however, we focus on just bar, line and circle graphs, we can probably take care of most situations. We should, however, understand the basic properties of each of those three types.

- Bar graphs are the best way to display raw quantities that can be arranged in any order.
- Line graphs are used to draw attention to changes in data over time.
- Pie charts should be used to display relative percentages of disjoint data.

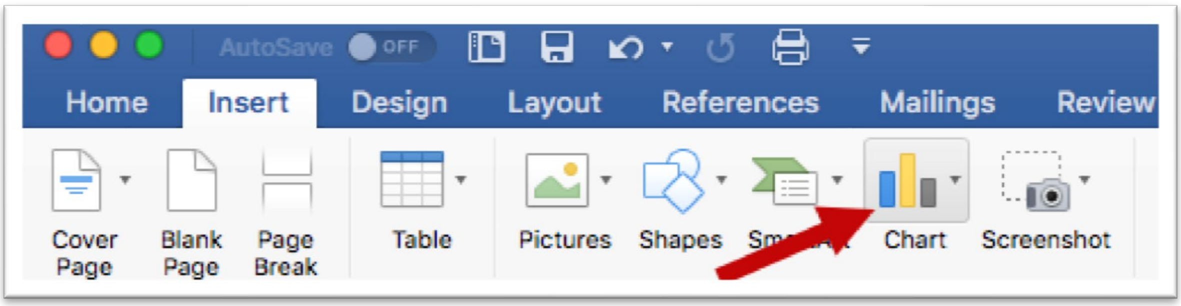
This does not mean we are limited to one specific type of graph or chart for a given set of data. When we are looking at the different colors of jellybeans in a bag, if we want to focus on the quantities, we can display the raw numbers in a bar chart. If we want to show half the bag contains red jellybeans, we may want to count all the beans and then display the data in a pie chart, with the "red" sector being 180°.

If we are looking at the number of apples sold by Bob's orchard, we can display the monthly total with a pictograph. Alternatively, if we want to emphasize the increasing (or decreasing) sales, we can use a line graph.

Essentially, the type of graph we choose will depend on the structure of the data set AND what we are trying to emphasize.

Inserting a Chart in MS Word

Attractive and colorful charts and graphs are easy to make using **Microsoft (MS) Word**, or any word processor. Although the directions may differ from program to program, to insert a chart in MS Word, select Chart on the Insert ribbon.

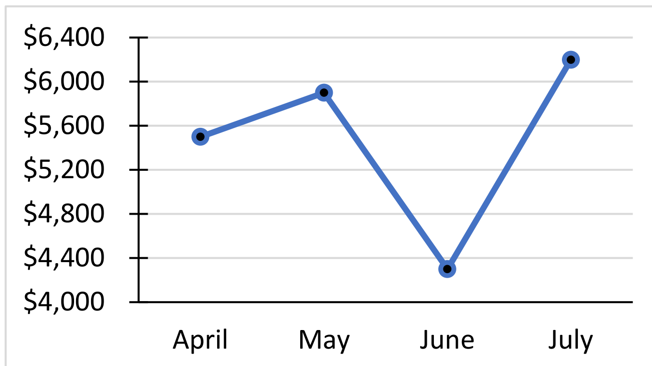


Once inserted, the chart type and options can be altered through the choices via different menus. Each type of chart will start you off with an example data set, which can easily be modified to suit your desires. Once done, they can easily be copy-and-pasted between documents. In fact, most of the charts and graphs found throughout this section were made with MS Word.

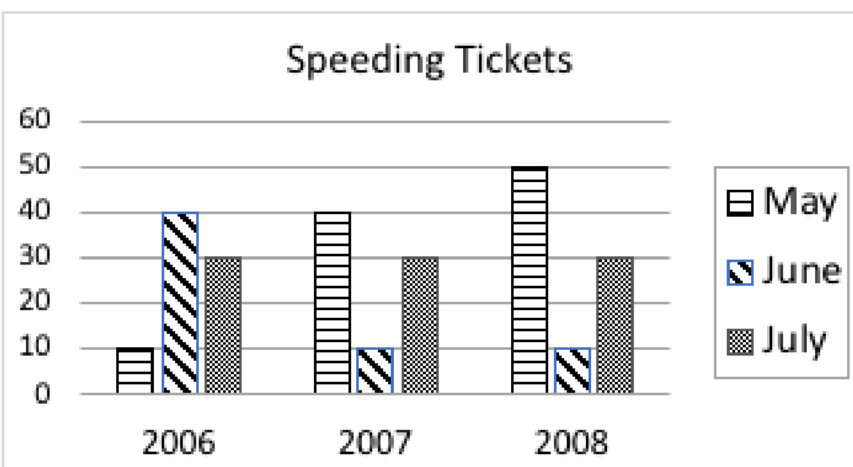
Section 4.5 Exercises

1. There are 34 students in Joe's math class, 27 in his English class, 22 in his history class, and 17 students in his philosophy class. Create a bar graph that represents this data.

2. The line graph displays the monthly profit made by a small hardware store over a four-month period last year. During which period did the monthly profit decline?



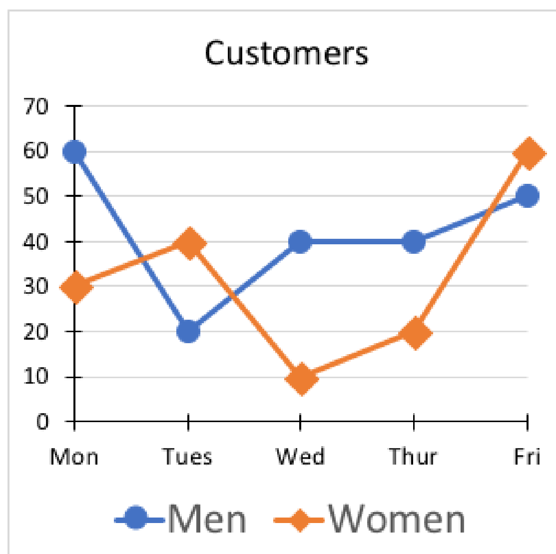
3. The given graph displays the number of speeding tickets issued by the Mayberry police department during the months of May, June and July of the years 2006, 2007, and 2008. How many tickets were issued in June of 2006? For the months shown on the graph, during which year was the most tickets issued?



4. A survey of college students, that asked them about the mode of transportation that they used to travel to campus, resulted in the following information: 120 students drove a car to school, 80 students rode a bike, 70 students took the bus, and 30 students walked to school. Find the number of degrees that should be given to each category, and create a pie chart that represents this data.

5. The students in Mrs. Cook's class were asked about their pets. 24 students said they have a dog, 16 have a cat, 8 students have a fish, 10 students have a bird, and 2 students said they have a pet turtle. Create a bar graph that represents this data.

6. Consider the clustered line graph representing the gender of the customers at a yogurt shop last week. How many women were there on Friday?



On which days were there more men than women?

How many customers were there on Tuesday?

7. In a Physics class, 4 students earned As, 9 earned Bs, 14 earned Cs, 7 got Ds, and 6 students were issued Fs. Find the number of degrees that should be given to each category, and create a pie chart that represents this data.

8. In 1920, the population of Fictionland was 3400. In 1940, the population was 2500. In 1960, the population was 4900. In 1980, the population was 6300. In 2000, the population rose to 5000. Create a line graph that represents this data.

9. Which kind of graph is best for displaying relative percentages of disjoint data?

10. A group of children purchased bags of jellybeans from a candy store. Later, they discovered 5 of the bags contained twelve jellybeans, 12 bags contained thirteen jellybeans, 9 bags contained fourteen jellybeans, and 4 bags had fifteen beans. Create a bar graph that represents this data.

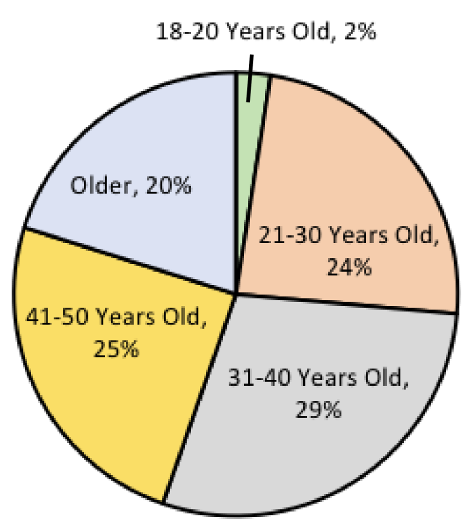
11. If the percent values for all the sectors of a pie chart are added together, what is the total?

12. Which kind of graph is best for showing comparison and including the raw data.

13. In 2014, 3800 students took English 101, and 2000 students took MATH 120. In 2015, 3500 students took English 101 and 2800 students took MATH 120. In 2016, 4000 students took English 101 and 3500 students took MATH 120. In 2017, 3800 students took English 101, and 4200 took MATH 120. Create a clustered line graph that represents this data.

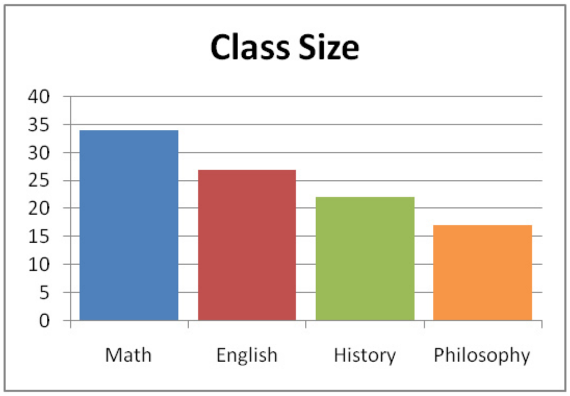
14. Which kind of graph is best for displaying the change in data over time?

15. The pie chart shown gives the percentage of voters in each of five different age groups. If there are 3,950,000 voters in the state, how many are in each age group?



Section 4.5 Exercise Solutions

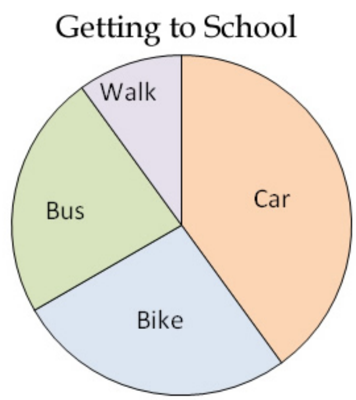
1. While we could arrange the classes in any order, one option is:



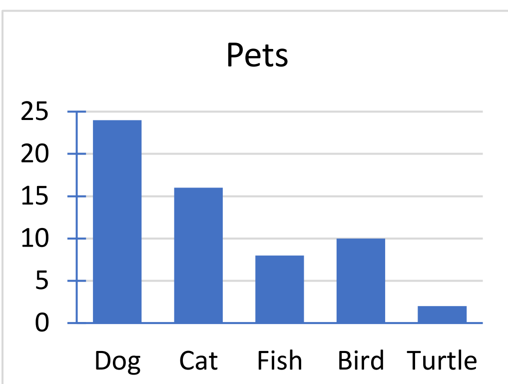
2. The period from May to June saw a profit decrease.

3. For the first question, there were 40 tickets issued in June of 2006. For the second question, add together the bar heights for the three-month period for each year to see the most tickets were issued in 2008.

4. First note that there are 300 students in the survey.
   Car = (120/300)(360°) = 144°
   Bike = (80/300)(360°) = 96°
   Bus = (70/300)(360°) = 84°
   Walk = (30/300)(360°) = 36°

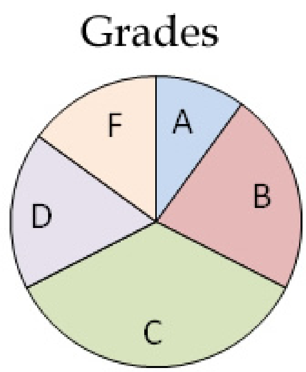


5.

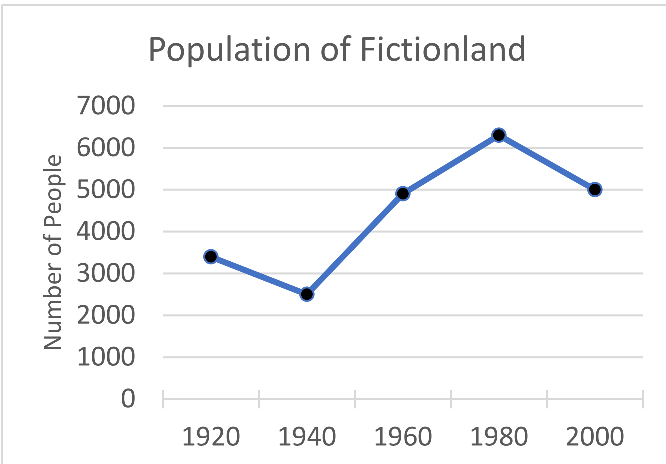


6. On Friday, there were 60 women.
   There were more men on Monday, Wednesday, and Thursday.
   On Tuesday, there were 60 customers.

7. First note that there are 40 students in the class.
   A = (4/40)(360°) = 36°
   B = (9/40)(360°) = 81°
   C = (14/40)(360°) = 126°
   D = (7/40)(360°) = 63°
   F = (6/40)(360°) = 54°

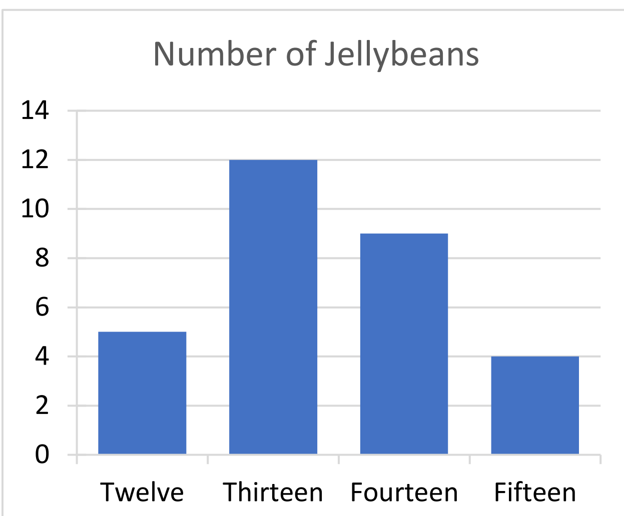


8.



9. A pie chart.

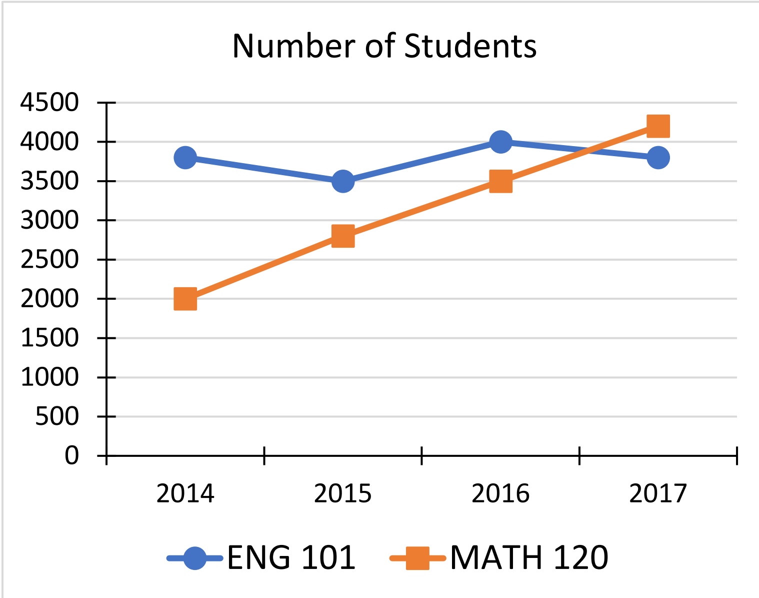
10.



11. 100%

12. A bar graph.

13.



14. A line graph.

15. 18-20 Year-Olds: (0.02)(3,950,000) = 79,000 voters
    21-30 Year-Olds: (0.24)(3,950,000) = 948,000 voters
    31-40 Year-Olds: (0.29)(3,950,000) = 1,145,500 voters
    41-50 Year-Olds: (0.25)(3,950,000) = 987,500 voters
    Older: (0.20)(3,950,000) = 790,000 voters

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